Annals of Mathematical Logic 10 (1976) 401–439

ALMOST-DISJOINT SETS, THE DENSE SET PROBLEM AND THE PARTITION CALCULUS *

James E. BAUMGARTNER
Department of Mathematics, Dartmouth College, Hanover, NH 03755, U.S.A.

Received 1 June 1972

Two sets A and B are said to be *almost disjoint* if the cardinality of $A \cap B$ is smaller than the cardinality of either A or B. In part, this paper is a sequel to the classical papers of Sierpiński [18] and Tarski [20] on almost-disjoint sets. The problems considered here are of the following form: Given a set X of specified cardinality κ, how large a collection F of pairwise almost-disjoint subsets of X can there be? This question can be modified by requiring that every member of F have a given cardinality μ, or by requiring that $A \cap B$ have cardinality smaller than a given cardinal ν whenever $A, B \in F$ and $A \neq B$.

In general, the answers to these questions depend on which axioms are taken for set theory. We have obtained a complete solution if the generalized continuum hypothesis is assumed. This completes the analysis begun by Tarski. If the generalized continuum hypothesis is not assumed, then the situation is much more complicated. We have some positive results which do not seem to follow from the results of Sierpiński and Tarski. Many of these are quite elementary. Using Cohen style independence techniques, we are frequently able to show that the positive results cannot be improved, but there are still some open problems.

A problem closely related to the almost-disjoint set problem is the following, which derives from a question of Malitz in [13]:

* Part of the material in this paper is contained in Chapter 3 of the author's Ph.D. Dissertation [2], prepared under the supervision of Professor Robert L. Vaught. A preliminary statement of the results in Section 3 was announced in [1].

Given a cardinal κ, for which cardinals λ do there exist totally ordered sets $(S,<)$ of power λ with dense subsets of power κ?
This problem can be modified by requiring the character of each element of S to be a given cardinal μ.

Independence results concerning this problem have been found by Mitchell [16]. It turns out that many of our positive results for the almost-disjoint set problem hold also for this problem, and some of our independence results imply the corresponding results of Mitchell.

It also turns out that some of the methods we use to deal with almost-disjoint sets can be used to obtain consistency results in the partition calculus. For example, we prove that the following are all (separately) consistent with $2^{\aleph_0} \geqslant \aleph_2$:

$$\binom{2^{\aleph_0}}{\aleph_1} \to \binom{2^{\aleph_0}}{\alpha}^{1,1}_{\aleph_0} \qquad \text{for all } \alpha < \omega_1 ,$$

$$\binom{2^{\aleph_0}}{\aleph_1} \not\to \binom{2}{\aleph_0}^{1,1}_{\aleph_0} ,$$

$$\binom{2^{\aleph_0}}{\aleph_1} \to \begin{pmatrix} 2^{\aleph_0} & 2^{\aleph_0} \\ \aleph_1 & \alpha \end{pmatrix}^{1,1} \qquad \text{for all } \alpha < \omega_1 ,$$

$$\binom{2^{\aleph_0}}{\aleph_1} \not\to \begin{pmatrix} \aleph_1 & \aleph_1 \\ \aleph_0 & \aleph_0 \end{pmatrix}^{1,1} ,$$

$$2^{\aleph_0} \to (2^{\aleph_0}, [\alpha, 2^{\aleph_0}])^2 \qquad \text{for all } \alpha < \omega_1 ,$$

$$2^{\aleph_0} \not\to (2^{\aleph_0}, [\aleph_0, 2])^2 .$$

(The consistency of the last proposition is an unpublished result of R. Laver, who uses a different method. Unfortunately, Laver's method does not seem to generalize.)

The paper is organized as follows. Notation and terminology are in Section 1. Section 2 contains a theorem (due mostly to Mitchell) giving alternate characterizations of the dense-set problem, and a number of elementary results, including the basic results of Sierpiński and Tarski, about both this problem and almost-disjoint sets.

In Section 3 we prove a simple combinatorial theorem and obtain as corollaries substantial improvements of the Sierpiński and Tarski results. For example, from Sierpinski's results it follows that if $2^{\aleph_0} = \aleph_1$ or $2^{\aleph_0} = 2^{\aleph_1}$, then there is a family of $2^{\aleph_1}$ pairwise almost-disjoint subsets of a set of power $\aleph_1$. We obtain the same conclusion from the weaker hypothesis that $2^{\aleph_0} = 2^{\aleph_1}$ or $2^{\aleph_0} < \aleph_{\omega_1}$. The result assuming the generalized continuum hypothesis is in this section also.

Section 4 contains several corollaries of an observation of Shelah. A typical theorem is that if there is an ordered set of power λ with a dense subset of power κ, then the same is true if κ and λ are replaced by κ^ρ and λ^ρ, where ρ is arbitrary.

The independence results are in Sections 5,6 and 7. A special case of Theorem 5.6 says that it is consistent with ZFC that $2^{\aleph_0} = \aleph_{\omega_1}$, $2^{\aleph_1} = \aleph_{\omega_1+1}$ and there is no family of $2^{\aleph_1}$ pairwise almost-disjoint subsets of a set of power $\aleph_1$. A special case of the main theorems of Section 6 says that both the following proposition and its negation are consistent with ZFC + $2^{\aleph_0} \geqslant \aleph_2$:

(1) *If A has power $\aleph_1$, then there is a family F of $\aleph_2$ subsets of A such that all $X \in F$ have power $\aleph_1$ and $X \cap Y$ is finite if $X, Y \in F$ and $X \neq Y$.*

In proving the consistency of the negation of (1), we prove the consistency with $2^{\aleph_0} \geqslant \aleph_2$ of the following proposition, which may be of independent interest:

(2) *There is a collection A of infinite sets of ordinals such that A has power $\aleph_1$ and for every cofinal subset B of ω_1 there is $X \in A$ with $X \subseteq B$.*

Section 7 contains the consistency results for the partition calculus.

1. Notation and Terminology

Our set-theoretical terminology is standard. We use $\alpha,\beta,\gamma,\delta,\xi,\eta,x,y$ for ordinals and $\kappa,\lambda,\mu,\nu,\rho,\sigma,\tau$ for (infinite) cardinals. Each ordinal is identified with the set of its predecessors. Since we assume the axiom of choice throughout, cardinals may be identified with initial ordinals. Thus $\aleph_\alpha = \omega_\alpha$ for all ordinals α.

If A is a set then $|A|$ denotes the cardinality of A and $\mathcal{P}(A)$ denotes the set of all subsets of A. If A and B are sets then ${}^AB$ denotes the set of all functions mapping A into B, and $A \times B$ denotes the cartesian product of A and B. The cartesian product of an indexed family $\langle A_i : i \in I\rangle$

of sets is denoted by $\Pi_{i \in I} A_i$. Of course, $f \in \Pi_{i \in I} A_i$ iff f is a function with domain I and $f(i) \in A_i$ for all $i \in I$.

If α is an ordinal then $\mathrm{cf}\alpha$ is the *cofinality* of α, i.e., the least ordinal β which can be mapped into a cofinal subset of α in an order-preserving fashion.

For cardinals κ and λ, $\kappa^{<\lambda} = \Sigma_{\mu < \lambda} \kappa^\mu$. The cardinal successor of κ is denoted by κ^+.

A subset A of a cardinal κ is *unbounded* if for every $\alpha < \kappa$ there is $\beta > \alpha$ with $\beta \in A$. A is *closed* if $\sup X \in A$ for every bounded set $X \subseteq A$. A is *stationary in* κ if A has non-empty intersection with every closed, unbounded subset of κ. A cardinal κ is *Mahlo* if the set of all strongly inaccessible cardinals less than κ is stationary in κ.

ZF is Zermelo–Fraenkel set theory. ZFC is ZF with the axiom of choice. GCH is the generalized continuum hypothesis. V = L is Gödel's axiom of constructibility.

If A and B are sets then $[A,B] = \{\{a,b\} : a \in A, b \in B, a \neq b\}$. Let $[A]^2 = [A,A]$, the set of two-element subsets of A. Let α,β,γ and δ be ordinals. The notation $\alpha \to (\beta,\gamma)^2$ means that for any $f : [\alpha]^2 \to 2$, there is a set $A \subseteq \alpha$ such that either A has order-type β and $f(x) = 0$ for all $x \in [A]^2$, or else A has order-type γ and $f(x) = 1$ for all $x \in [A]^2$. The set A is said to be *homogeneous for f*. The notation $\alpha \to (\beta,[\gamma,\delta])^2$ means that if f: $[\alpha]^2 \to 2$ then either there is $A \subseteq \alpha$ of order-type β with $f(x) = 0$ for all $x \in [A]^2$, or else there are $C,D \subseteq \alpha$ of type γ and δ respectively such that $f(x) = 1$ for all $x \in [C,D]$. Since cardinals are particular examples of ordinals, this notation makes sense for cardinals also. Now let $\kappa,\lambda,\mu_\xi,\nu_\xi$ be cardinals for all $\xi < \rho$. The notation

$$\begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \to \begin{pmatrix} \mu_\xi \\ \nu_\xi \end{pmatrix}_\rho^{1,1}$$

means that if f: $\kappa \times \lambda \to \rho$, then there are $A \subseteq \kappa$, $B \subseteq \lambda$ and $\xi < \rho$ such that $|A| = \mu_\xi$, $|B| = \nu_\xi$ and $f(\alpha,\beta) = \xi$ for all $(\alpha,\beta) \in A \times B$. We write

$$\begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \to \begin{pmatrix} \mu_0 & \mu_1 \\ \nu_0 & \nu_1 \end{pmatrix}^{1,1} \quad \text{if } \rho = 2\,,$$

$$\begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \to \begin{pmatrix} \mu \\ \nu \end{pmatrix}_\rho^{1,1} \qquad \text{if } \mu_\xi = \mu,\ \nu_\xi = \nu$$

for all $\xi < \rho$. We write

$$\binom{\kappa}{\lambda} \to \left(\binom{\mu}{\nu}\binom{\mu'}{\nu'}_{\rho'}\right)^{1,1}$$

if $\rho = 1 + \rho'$, $\mu_0 = \mu$, $\nu_0 = \nu$, and $\mu_{1+\xi} = \mu'$ and $\nu_{1+\xi} = \nu'$ for all $\xi < \rho'$. The negations of all these propositions are indicated by striking out the arrow

A partially ordered set $(T, \leqslant)$ is a *tree* if for all $t \in T$, $\{s \in T: s \leqslant t\}$ is well-ordered by $\leqslant$. The *level* of $t \in T$, $l(t)$, is the order-type of $\{s \in T: s < t\}$. The *height* of the tree $(T, \leqslant)$ is $\sup\{l(t) + 1: t \in T\}$. A *branch* is a maximal totally ordered subset of T. The *length* of a branch is its order-type.

We assume the reader is familiar with the notions of ultrafilters and untrapowers (see [3]). We mention that an ultrafilter D on a set I is *regular* if there is $E \subseteq D$ such that $|E| = |I|$ and $\bigcap X = 0$ for every infinite $X \subseteq E$.

We also assume the reader is familiar with the theory of forcing and generic sets (see [10] and [19]). Let $(P, \leqslant)$ be a partially ordered set. Two elements $p, q \in P$ are *compatible* if there is $r \in P$ with $r \leqslant p$ and $r \leqslant q$; otherwise p and q are *incompatible*. A set $D \subseteq P$ is *dense in P* if for every $p \in P$ there is $q \in D$ with $q \leqslant p$; D is *dense below p* if for every $q \leqslant p$ there is $r \in D$ with $r \leqslant q$.

P has the *κ-chain condition* provided that every set of pairwise incompatible elements of P has cardinality smaller than κ. The $\aleph_1$-chain condition is usually called the *countable chain condition*. P is *κ-complete* provided that for all $\alpha < \kappa$, if $\langle p_\beta : \beta < \alpha\rangle$ is a decreasing sequence of members of P then there is $p \in P$ such that $p \leqslant p_\beta$ for all $\beta < \alpha$.

Let $\mathfrak{M}$ be a countable transitive model of ZFC, and let $(P, \leqslant) \in \mathfrak{M}$. A set $G \subseteq P$ is *P-generic over* $\mathfrak{M}$ if

(1) if $p, q \in G$, then p and q are compatible;

(2) if $p \in G$ and $p \leqslant q$, then $q \in G$;

(3) if D is dense in P and $D \in \mathfrak{M}$, then $G \cap D \neq 0$.

If G is P-generic over $\mathfrak{M}$ then $\mathfrak{M}[G]$ is the smallest transitive model of ZFC such that $\mathfrak{M} \subseteq \mathfrak{M}[G]$ and $G \in \mathfrak{M}[G]$. The forcing relation $\Vdash$ between members of P and sentences of the language of forcing may be defined by letting $p \Vdash \varphi$ iff for every G, if G is P-generic over $\mathfrak{M}$ and $p \in G$, then φ is true in $\mathfrak{M}[G]$. An important fact is that if φ is true in $\mathfrak{M}[G]$, then, $p \Vdash \varphi$ for some $p \in G$. Every member of $\mathfrak{M}[G]$ is denoted by some term of the language of forcing.

As is customary, if τ is a term of the language of set theory then $\tau^{\mathfrak{M}}$ denotes the relativization of that term to the model $\mathfrak{M}$. For example, $\omega_1^{\mathfrak{M}}$ refers to the least uncountable cardinal *as defined in* $\mathfrak{M}$.

2.

Let κ, λ, μ and ν be infinite cardinals. We write $A(\kappa,\lambda,\mu,\nu)$ to mean that there exists a family F such that

(1) $F \subseteq \mathcal{P}(\kappa)$

(2) $|F| = \lambda$

(3) $|X| = \mu$ for all $X \in F$

(4) $|X \cap Y| < \nu$ if $X, Y \in F$ and $X \neq Y$.

We use $A(\kappa,\lambda,\mu)$ as an abbreviation for $A(\kappa,\lambda,\mu,\mu)$. We write $A(\kappa,\lambda)$ to mean that there is a family F satisfying (1), (2) and

(5) if $X, Y \in F$ and $X \neq Y$, then $|X \cap Y| < |X|, |Y|$ (i.e., the elements of F are pairwise almost-disjoint).

Since all these propositions are trivial if $\lambda \leqslant \kappa$, $\kappa < \mu$ or $\mu < \nu$, we will only be interested in the case when $\nu \leqslant \mu \leqslant \kappa < \lambda$.

If F satisfies (1), (2), (3) and (4), then we refer to F as a (κ,λ,μ,ν)-*family*. We define (κ,λ,μ)-families and (κ,λ)-families similarly. If $|X| = \kappa$, $F \subseteq \mathcal{P}(X)$ and F satisfies (2), (3) and (4), then we say that F is a (κ,λ,μ,ν)-family of subsets of X, and so on.

Now let S be a totally ordered set and let $U \subseteq S$. We say that U is *weakly dense in* S if $s, t \in S$, $s < t$ implies that there is $u \in U$ with $s \leqslant u \leqslant t$. Of course, U is *dense in* S if $s, t \in S$, $s < t$ imples that there is $u \in U$ with $s < u < t$. (Notice that this usage of the word "dense" is entirely different from the usage in the theory of forcing and generic sets.)

We write $D(\kappa,\lambda)$ to mean that there exist S and U such that

(6) $|U| = \kappa$ and $|S| = \lambda$,

(7) S is totally ordered and U is weakly dense in S.

It is easy to see that $D(\kappa,\lambda)$ holds iff there are S and U satisfying (6) and

(8) S is totally ordered and U is dense in S.

If S and U satisfy (6) and (7) then we refer to (S, U) as a (κ,λ)-*pair*.

If S is totally ordered and $s \in S$, then the *left character of* s is the smallest cardinal μ such that there exists a strictly increasing sequence $s_\alpha \in S$, $\alpha < \mu$, cofinal in $\{t \in S\colon t < s\}$. Note that if $\mu \neq 1$ (which will always happen if S is densely ordered) and U is weakly dense in S, then all the s_α may be chosen from U. The *right character of* s is the smallest μ such that there exists a strictly decreasing sequence s_α, $\alpha < \mu$, co-initial in $\{t \in S: s < t\}$. The *character of* s, $\chi(s)$, is the smaller of the left and right characters of s. Clearly $\chi(s)$ is always regular.

We write $D(\kappa,\lambda,\mu)$ to mean that there exist S and U satisfying (6), (7) and

(9) $\chi(s) = \mu$ for all $s \in S - U$.

If S and U satisfy (6), (7) and (9), then (S, U) is a (κ,λ,μ)*-pair*.

Before we proceed to the statement of some easy results about the properties A and D, it will be convenient to have the following characterization of property D. The equivalence of (i) and (ii) in part (b) of Theorem 2.1 is due to W. Mitchell. The equivalence of (i) and (iii) is due independently to W. Mitchell and the author.

Theorem 2.1. *Let $\mu \leqslant \kappa \leqslant \lambda$, and assume μ is regular.*

(a) *$D(\kappa,\lambda,\mu)$ holds iff there is a tree of height μ and cardinality $\leqslant \kappa$ with at least λ branches of length μ.*

(b) *The following are equivalent:*

(i) *$D(\kappa,\lambda)$*

(ii) *There is a tree of height $\leqslant \kappa$ and cardinality $\leqslant \kappa$ with at least λ branches*

(iii) *There is a family F of subsets of κ such that $|F| = \lambda$ and F is totally ordered by inclusion.*

Proof. If $\kappa = \lambda$ then the theorem is trivial. Assume $\kappa < \lambda$.

(a) Assume $D(\kappa,\lambda,\mu)$ and let (S, U) be a (κ,λ,μ)-pair. Let $\prec$ be a well-ordering of U in type κ. For each $s \in S - U$, we define an ordinal $\delta_s \leqslant \kappa$ and a function $u_s\colon \delta_s \to U$ by induction as follows. If α is even, let $u_s(\alpha)$ be the $\prec$-least member of U such that $u_s(\alpha) < s$ and $u_s(\beta) < u_s(\alpha)$ for all even $\beta < \alpha$, if such a member exists; if no such member exists, let $\delta_s = \alpha$. If α is odd, let $u_s(\alpha)$ be the $\prec$-least member of U such that $s < u_s(\alpha)$ and $u_s(\alpha) < u_s(\beta)$ for all odd $\beta < \alpha$, if such a member exists; if no such member exists, let $\delta_s = \gamma$, where $\alpha = \gamma + 1$. It is easy to see that δ_s is a limit ordinal $\leqslant \kappa$ of cofinality $\chi(s) = \mu$.

Furthermore, we assert that there are at most κ functions of the form $u_s | \gamma$, where $s \in S - U$ and $\gamma < \delta_s$. Fix $s \in S - U$ and $\gamma < \delta_s$. Then we may find $u_1 \in \{u \in U: u_s(\alpha) < u < s$ for all even $\alpha < \gamma\}$ and $u_2 \in \{u \in U: s < u < u_s(\alpha)$ for all odd $\alpha < \gamma\}$. But then $\langle u_s(\alpha)\colon \alpha$ even, $\alpha < \gamma\rangle$ is definable from u_1 in exactly the same way it was defined from s, and the same holds for $\langle u_s(\alpha)\colon \alpha$ odd, $\alpha < \gamma\rangle$ and u_2. This establishes the assertion since $|U| = \kappa$ and $\gamma < \delta_s \leqslant \kappa$.

Now for each $\delta \leqslant \kappa$ with $\mathrm{cf}\delta = \mu$, let $S_\delta = \{s \in S - U\colon \delta_s = \delta\}$ and let δ_α, $\alpha < \mu$, be an increasing sequence of ordinals with limit δ. Let $T_\delta = \{u_s | \delta_\alpha\colon \alpha < \mu, s \in S_\delta\}$. Then $|T_\delta| \leqslant \kappa$ and T_δ is a tree of height μ when partially ordered by inclusion. Since $\{u_s|\delta_\alpha\colon \alpha < \mu\}$ is a branch through T_δ for each $s \in S_\delta$, T_δ has at least $|S_\delta|$ branches. Finally, Since $S - U = \bigcup_{\delta \leqslant \kappa} S_\delta$ it is easy to see that the trees T_δ can be combined into a single tree of height μ and cardinality $\leqslant \kappa$, and with at least $|S - U| = \lambda$ branches.

For the converse, assume $(T, \leqslant_T)$ is a tree of height μ and cardinality $\leqslant \kappa$ and let S be a set of branches through T of length μ with $|S| = \lambda$. Let $\prec$ be an arbitrary total ordering of T. Let S be totally ordered by the lexicographical ordering, i.e., set $B_1 < B_2$ iff $t_1 \prec t_2$, where t_1 is the $\leqslant_T$-least element of $B_1 - B_2$ and t_2 is the $\leqslant_T$-least element of $B_2 - B_1$. For each $t \in T$, choose $B_t \in S$ such that $t \in B_t$, if such B_t exists. If there is a $<$-minimal B such that $t \in B$, then we assume $B = B_t$. Let $U \subseteq S$ be such that $U \supseteq \{B_t\colon t \in T\}$ and $|U| = \kappa$. It is easy to check that U is weakly dense in S and $\chi(B) = \mu$ for all $B \in S - U$. Hence $D(\kappa,\lambda,\mu)$ holds.

(b) (i) implies (ii). Let (S, U) be a (κ,λ)-pair. As we remarked earlier, we may assume that U is dense in S. Hence $\chi(s) \geqslant \omega$ for all $s \in S$. For each $\mu \leqslant \kappa$, let

$$S_\mu = \{s \in S - U\colon \chi(s) = \mu\}\ .$$

Then $(S_\mu \cup U, U)$ is a $(\kappa, |S_\mu|, \mu)$-pair, so by part (a) there is a tree of height μ and cardinality $\leqslant \kappa$ with at least $|S_\mu|$ branches. But all these trees can be combined into a single tree of height $\leqslant \kappa$ and cardinality κ with at least $|\bigcup_{\mu \leqslant \kappa} S_\mu| = \lambda$ branches.

(ii) implies (i). Given a tree as in (ii), sinply order the branches lexicographically as in the proof of part (a).

(i) implies (iii). Let (S, U) be a (κ,λ)-pair. For $s \in S$ let $X_s = \{u \in U\colon u < s\}$, the lower Dedekind cut determined by s. Then $F = \{X_s\colon s \in S\}$ satisfies (iii)

(iii) implies (i). Suppose F satisfies (iii). For each $\alpha \in \kappa$ let $A_\alpha = \bigcup\{A \in F\colon \alpha \notin A\}$. It is easily checked that $S = F \cup \{A_\alpha\colon \alpha \in \kappa\}$ is totally ordered by $\subseteq$. Let $U \subseteq S$ be such that $U \supseteq \{A_\alpha\colon \alpha \in \kappa\}$ and $|U| = \kappa$. We claim (S, U) is a (κ,λ)-pair. It will suffice to show that if $A, B \in S - U$, $A \subseteq B$ and $A \neq B$, then for some α, $A \subseteq A_\alpha \subseteq B$. But any $\alpha \in B - A$ will work. $\square$

Part (d) of the following theorem is essentially due to Mitchell.

Theorem 2.2. *Let* $\nu \leqslant \mu \leqslant \kappa \leqslant \lambda$.

(a) $A(\kappa,\lambda,\mu,\nu) \Rightarrow A(\kappa,\lambda,\mu) \Rightarrow A(\kappa,\lambda)$ *and* $D(\kappa,\lambda,\mu) \Rightarrow D(\kappa,\lambda)$.

(b) *Let* $\nu' \leqslant \mu' \leqslant \kappa' \leqslant \lambda'$. *If* $\kappa \leqslant \kappa'$, $\lambda' \leqslant \lambda$, $\mu' \leqslant \mu$ *and* $\nu \leqslant \nu'$, *then*

$$A(\kappa,\lambda,\mu,\nu) \Rightarrow A(\kappa',\lambda',\mu',\nu'),$$

$$A(\kappa,\lambda,\mu) \Rightarrow A(\kappa',\lambda',\mu'),$$

$$A(\kappa,\lambda) \Rightarrow A(\kappa',\lambda'),$$

$$D(\kappa,\lambda,\mu) \Rightarrow D(\kappa',\lambda',\mu),$$

$$D(\kappa,\lambda) \Rightarrow D(\kappa',\lambda').$$

(c) *Assume* $\lambda = \Sigma_{\alpha<\kappa} \lambda_\alpha$. *Then*

$$(\forall\alpha < \kappa)\, A(\kappa,\lambda_\alpha,\mu,\nu) \Rightarrow A(\kappa,\lambda,\mu,\nu),$$

$$(\forall\alpha < \kappa)\, A(\kappa,\lambda_\alpha,\mu) \Rightarrow A(\kappa,\lambda,\mu),$$

$$(\forall\alpha < \kappa)\, A(\kappa,\lambda_\alpha) \Rightarrow A(\kappa,\lambda),$$

$$(\forall\alpha < \kappa)\, D(\kappa,\lambda_\alpha,\mu) \Rightarrow D(\kappa,\lambda,\mu),$$

$$(\forall\alpha < \kappa)\, D(\kappa,\lambda_\alpha) \Rightarrow D(\kappa,\lambda).$$

(d) $D(\kappa,\lambda,\mu) \Rightarrow A(\kappa,\lambda,\mu)$ *and* $D(\kappa,\lambda) \Rightarrow A(\kappa,\lambda)$.

(e) $A(\kappa,(2^\kappa)^+)$ *and* $D(\kappa,(2^\kappa)^+)$ *are false. If* κ *is singular, then* $D(\kappa,(\kappa^{<\kappa})^+)$ *is false.*

Proof. (a) and (b) are trivial.

(c) Let K_α, $\alpha < \kappa$, be a sequence of disjoint subsets of κ, each of cardinality κ. Let F_α be a $(\kappa,\lambda_\alpha,\mu,\nu)$-family of subsets of K_α. Then $\bigcup_{\alpha<\kappa} F_\alpha$ is a (κ,λ,μ,ν)-family. The other assertions are all proved similarly.

(d) We may assume $\kappa < \lambda$. Let (S,U) be a (κ,λ,μ)-pair. For each $s \in S - U$, let $\langle u_s(\alpha)\colon \alpha < \mu\rangle$ be either a strictly increasing or strictly decreasing sequence of elements of U with limit s. Let $U_s = \{u_s(\alpha)\colon \alpha < \mu\}$. Then $\{U_s\colon s \in S - U\}$ is a (κ,λ,μ)-family. Now let (S,U) be a (κ,λ)-pair, and assume U is dense in S. For each $\mu \leqslant \kappa$, let $S_\mu = \{s \in S - U: \chi(s) = \mu\}$. As we have seen, for each $\mu \leqslant \kappa$ we may obtain a $(\kappa, |S_\mu|, \mu)$-family. But this implies the existence of a $(\kappa, \Sigma_{\mu<\kappa} |S_\mu|)$-family by (c).

Part (d) may also be proved using the characterization of property D in terms of trees.

(e) The first sentence is trivial. Suppose κ is singular and (S, U) is a (κ,λ)-pair with U dense in S. For each $\mu < \kappa$ there are at most κ^μ elements of $S - U$ of character μ, since each such element may be identified with a μ-sequence of elements of U. Hence there are at most $\kappa^{<\kappa}$ elements of $S - U$ of character $<\kappa$. But since κ is singular, every element of $S - U$ has character $<\kappa$. Hence $\lambda \leqslant \kappa^{<\kappa}$. □

The following results (Theorem 2.3–Theorem 2.7), are due to Tarski [20] for property A.

Theorem 2.3. *Let κ and λ be cardinals (λ may be finite), and let μ be the least cardinal such that $\kappa < \lambda^\mu$. Then $A(\kappa, \lambda^\mu, \mu)$, $A(\kappa, \lambda^\mu, \mathrm{cf}\mu)$ and $D(\kappa, \lambda^\mu, \mathrm{cf}\mu)$ hold.*

Proof. Let $T = \bigcup_{\alpha<\mu} {}^\alpha\lambda$. T is a tree when ordered by inclusion. By assumption T has $\Sigma_{\alpha<\mu} \lambda^{|\alpha|} \leqslant \kappa$ elements. Of course, T has height μ and λ^μ branches of length μ. Moreover, these branches are almost disjoint, so they form a $(\kappa, \lambda^\mu, \mu)$-family of subsets of T. If μ is regular we are done. If μ is singular, let μ_α, $\alpha < \mathrm{cf}\mu$, be an increasing sequence of ordinals with limit μ. Let $T' = \bigcup_{\alpha<\mathrm{cf}\mu} {}^{(\mu_\alpha)}\lambda$ and repeat the argument with T' instead of T. □

This is essentially Sierpinski's proof in [18] that $A(\kappa, \kappa^+)$ holds for all κ.

Corollary 2.4. *Let κ, λ, μ be as in Theorem 2.3. Then $A(\kappa, \lambda^\mu)$ and $D(\kappa, \lambda^\mu)$ hold.*

Corollary 2.5. *Assume* GCH. *Then $D(\kappa, \kappa^+, \mathrm{cf}\kappa)$, $A(\kappa, \kappa^+, \kappa)$ and $A(\kappa, \kappa^+, \mathrm{cf}\kappa)$ hold for all κ.*

Proof. Assuming GCH, we have $2^{<\kappa} = \kappa$. □

Corollary 2.6. *If κ is a strong limit cardinal, then $D(\kappa, \kappa^+, \mathrm{cf}\kappa)$, $A(\kappa, \kappa^+, \kappa)$ and $A(\kappa, \kappa^+, \mathrm{cf}\kappa)$ hold.*

Theorem 2.7. *If $A(\kappa,\lambda,\mu,\nu)$ holds, then $\lambda \leqslant \kappa^\nu$. More generally, if F is a (κ,λ)-family and $|X \cap Y| < \nu$ whenever $X, Y \in F$ and $X \neq Y$, then $\lambda \leqslant \kappa^\nu$. Hence if $D(\kappa,\lambda,\mu)$ holds, then $\lambda \leqslant \kappa^\mu$.*

Proof. Let F be as indicated. For each $X \in F$, let X' be a subset of X of power ν, if $|X| \geqslant \nu$; otherwise let $X' = X$. If $\lambda > \kappa^\nu$ then there are X, $Y \in F$ such that $X \neq Y$ and $X' = Y'$. But then $|X \cap Y| \geqslant \nu$, contradiction. □

All the results in Sierpinski [18] and Tarski [20] follows from Theorems 2.3 and 2.7.

It is interesting to note that Corollary 2.5 for property A is provable without GCH. However, Mitchell [16] has shown that $D(\kappa,\kappa^+,\kappa)$ is not provable without GCH.

Theorem 2.8. *For all κ, $A(\kappa,\kappa^+,\kappa)$ holds. If κ is singular, then $A(\kappa,\kappa^+, \mathrm{cf}\kappa)$ holds also.*

Proof. Assume κ is regular. It will suffice to show that if $\langle X_\alpha : \alpha < \kappa\rangle$ is a sequence of pairwise almost-disjoint subsets of κ, each of power κ, then there exists $X \subseteq \kappa$ such that $|X| = \kappa$ and $|X \cap X_\alpha| < \kappa$ for all $\alpha < \kappa$. Simply choose distinct $\gamma_\alpha \in \kappa - \bigcup_{\beta<\alpha} X_\beta$ for $\alpha < \kappa$ and let $X = \{\gamma_\alpha : \alpha < \kappa\}$.

Now assume κ is singular. Suppose $\langle X_\alpha : \alpha < \kappa\rangle$ is a sequence of pairwise almost-disjoint subsets of κ, each of power $\mathrm{cf}\kappa$. We will find $X \subseteq \kappa$ such that $|X| = \mathrm{cf}\kappa$ and $|X \cap X_\alpha| < \mathrm{cf}\kappa$ for each $\alpha < \kappa$. Let κ_α, $\alpha < \mathrm{cf}\kappa$, be an increasing sequence of cardinals with limit κ. Now choose distinct $\gamma_\alpha \in \kappa - \bigcup_{\beta<\kappa_\alpha} X_\beta$ for $\alpha < \mathrm{cf}\kappa$, and let $X = \{\gamma_\alpha : \alpha < \mathrm{cf}\kappa\}$. $A(\kappa,\kappa^+\kappa)$ may be proved similarly, or one may apply the following theorem. □

Theorem 2.9. *Suppose $\mu \leqslant \mu' \leqslant \kappa < \lambda$ and $\mathrm{cf}\mu' = \mu$. Then $A(\kappa,\lambda,\mu)$ implies $A(\kappa,\lambda,\mu')$.*

Proof. Let F be a (κ,λ,μ)-family. We may assume that each member of F has order-type μ (with respect to the usual ordering on κ). For each $\alpha \leqslant \kappa$, let $F_\alpha = \{X \in F : \sup X = \alpha\}$. By Theorem 2.2(c), it will suffice to prove $A(\kappa, |F_\alpha|, \mu')$ for all $\alpha \leqslant \kappa$. Fix $\alpha \leqslant \kappa$ such that $F_\alpha \neq 0$. Then $\mathrm{cf}\alpha = \mu$, so let $\langle \alpha_\xi : \xi < \mu\rangle$ be a strictly increasing sequence of ordinals with limit α. Let $\langle \mu'_\xi : \xi < \mu\rangle$ be a strictly increasing sequence of cardinals with limit μ'. Let $\langle Y_\eta : \eta < \alpha\rangle$ be a sequence of pairwise disjoint subsets of κ such that $|Y_\eta| = \mu'_\xi$ if $\alpha_\xi < \eta < \alpha_{\xi+1}$. Given $X \in F_\alpha$, let $X' = \bigcup\{Y_\eta : \eta \in X\}$. Then $\{X' : X \in F_\alpha\}$ is a $(\kappa_\alpha, |F_\alpha|, \mu')$-family, since each $X \in F_\alpha$ is cofinal in α. □

Theorem 2.10. *Suppose $A(\kappa,\lambda,\mu,\nu)$ holds and for each $\alpha < \lambda$, $A(\mu,\lambda_\alpha,\mu',\nu)$ holds. Then $A(\mu, \Sigma_{\alpha<\lambda}\lambda_\alpha, \mu',\nu)$ holds.*

Proof. Let $\{X_\alpha : \alpha < \lambda\}$ be a (κ,λ,μ,ν)-family, and for each $\alpha < \lambda$ let F_α be a $(\mu,\lambda_\alpha,\mu',\nu)$-family of subsets of X_α. Then $\bigcup_{\alpha<\lambda} F_\alpha$ is a $(\mu, \Sigma_{\alpha<\lambda}\lambda_\alpha, \mu',\nu)$-family. □

Corollary 2.11. *If λ is the least cardinal for which $A(\kappa,\lambda,\kappa)$ is false, then λ is regular.*

Corollary 2.11 for property D is not provable in ZFC, as is shown by Mitchell [16].

3.

In this section we prove a simple combinatorial theorem and use it to obtain more results concerning the properties A and D. In particular, we are able to determine completely when $A(\kappa,\lambda,\mu,\nu)$ and $D(\kappa,\lambda,\mu)$ hold if GCH is assumed. We also prove that properties A and D hold in many situations not covered by the results in the previous section.

Theorem 3.1. *Suppose $F \subseteq \mathcal{P}(\kappa)$, $|F| = \lambda$ and $|X| = \mu$ for all $X \in F$. If there is no $C \subseteq \kappa$ such that $|C| < \kappa$ and $|\{X \in F : |X \cap C| = \mu\}| = \lambda$, then $\mathrm{cf}\kappa = \mathrm{cf}\lambda$ or $\mathrm{cf}\kappa = \mathrm{cf}\mu$.*

Proof. Assume $\mathrm{cf}\kappa \neq \mathrm{cf}\lambda$ and $\mathrm{cf}\kappa \neq \mathrm{cf}\mu$. Since $\mathrm{cf}\kappa \neq \mathrm{cf}\mu$, for every $X \in F$ there is $\alpha_X < \kappa$ such that $|X \cap \alpha_X| = \mu$. Since $\mathrm{cf}\kappa \neq \mathrm{cf}\lambda$, there is $\alpha < \kappa$ such that $|\{X \in F : \alpha_X \leqslant \alpha\}| = \lambda$. But then $|\{X \in F : |X \cap \alpha| = \mu\}| = \lambda$, contradiction. □

Corollary 3.2. *If κ is the least cardinal such that there exists a tree of height μ and cardinality κ with at least λ branches of length μ, then $\mathrm{cf}\kappa = \mathrm{cf}\lambda$ or $\mathrm{cf}\kappa = \mathrm{cf}\mu$. Hence if κ is the least cardinal such that $D(\kappa,\lambda,\mu)$ holds, then $\mathrm{cf}\kappa = \mathrm{cf}\lambda$ or $\mathrm{cf}\kappa = \mathrm{cf}\mu$ $(= \mu)$.*

Proof. Let T be a tree of height μ and cardinality κ, and for each $\alpha < \lambda$, let B_α be a branch of T of length μ. If $\mathrm{cf}\kappa \neq \mathrm{cf}\lambda$, $\mathrm{cf}\mu$, then by applying Theorem 3.1 to $F = \{B_\alpha : \alpha < \lambda\}$, we obtain $C \subseteq T$ such that $|C| < \kappa$ and C, considered as a subtree of T, has at least λ branches of length μ, a contradiction. □

Corollary 3.3. *Let $\nu \leqslant \mu \leqslant \lambda$. If κ is the least cardinal such that $A(\kappa,\lambda,\mu,\nu)$ holds, then $\mathrm{cf}\kappa = \mathrm{cf}\lambda$ or $\mathrm{cf}\kappa = \mathrm{cf}\mu$.*

Theorem 3.4. (GCH) *Assume $\nu \leqslant \mu \leqslant \kappa$. Then*
(a) *$A(\kappa,\kappa^+,\mu,\nu)$ holds iff $\mu = \nu$ and $\mathrm{cf}\mu = \mathrm{cf}\kappa$,*
(b) *$D(\kappa,\kappa^+,\mu)$ holds iff $\mu = \mathrm{cf}\kappa$.*

Proof. Suppose $\nu < \mu$. Then there is μ' such that $\nu \leqslant \mu' \leqslant \mu$ and $\mathrm{cf}\mu' \neq \mathrm{cf}\kappa$ By Theorem 2.2(b), if $A(\kappa,\kappa^+,\mu,\nu)$ holds, then so does $A(\kappa,\kappa^+,\mu',\nu)$. But then, by Corollary 3.3, κ is not the least cardinal ρ such that $A(\rho,\kappa^+,\mu',\nu)$ holds, contradicting GCH by Theorem 2.2(e). Hence if $\nu < \mu$ then $A(\kappa,\kappa^+,\mu,\nu)$ is false.

Now assume $\mu = \nu$. By Corollary 2.5, $A(\kappa,\kappa^+,\kappa)$, $A(\kappa,\kappa^+,\mathrm{cf}\kappa)$ and $D(\kappa,\kappa^+,\mathrm{cf}\kappa)$ hold, so by Theorem 2.9, $A(\kappa,\kappa^+,\mu)$ holds whenever $\mathrm{cf}\mu = \mathrm{cf}\kappa$. Hence it will suffice to show that if $\mathrm{cf}\mu \neq \mathrm{cf}\kappa$, then $A(\kappa,\kappa^+,\mu)$ is false. But if $\mathrm{cf}\mu \neq \mathrm{cf}\kappa$ and $A(\kappa,\kappa^+,\mu)$ holds then by Corollary 3.3, κ is not the least cardinal ρ satisfying $A(\rho,\kappa^+,\mu)$, and this contradicts GCH. □

Theorem 3.4(a) answers some questions raised in Tarski [20]. If the trivial cases are added to Theorem 3.4, one obtains a complete characterization of properties A and D under the assumption of GCH.

Theorem 3.5. *Let κ,λ,λ' and μ be cardinals (λ may be finite), and let $\kappa = \aleph_\alpha$. If $\lambda^{<\mu} < \aleph_{\alpha+\mathrm{cf}\mu}$, $\lambda^{<\mu} < \lambda^\mu$, $\lambda' \leqslant \lambda^\mu$ and either $\mathrm{cf}\lambda' \leqslant \kappa$ or $\mathrm{cf}\lambda' > \lambda^{<\mu}$, then $A(\kappa,\lambda',\mathrm{cf}\mu)$ and $D(\kappa,\lambda',\mathrm{cf}\mu)$ hold. (Therefore $A(\kappa,\lambda',\mu')$ and $D(\kappa,\lambda',\mathrm{cf}\mu)$ hold for all $\lambda' < \lambda^\mu$ and all $\mu' \leqslant \kappa$ with $\mathrm{cf}\mu' = \mathrm{cf}\mu$). If in addition $\mu = \kappa$, then $A(\kappa,\lambda^\mu,\mu)$ (i.e., $A(\kappa, 2^\kappa, \kappa)$) holds also.*

Proof. By Theorem 2.3, $A(\lambda^{<\mu}, \lambda^\mu, \mathrm{cf}\mu)$ and $D(\lambda^{<\mu}, \lambda^\mu, \mathrm{cf}\mu)$ hold. Let ρ be the least cardinal such that $D(\rho,\lambda',\mathrm{cf}\mu)$ holds. Then $\rho \leqslant \lambda^{<\mu}$. By Corollary 3.2, $\mathrm{cf}\rho = \mathrm{cf}\lambda'$ or $\mathrm{cf}\rho = \mathrm{cf}\mu$. But this is impossible unless $\rho \leqslant \kappa$. Hence $D(\kappa,\lambda',\mathrm{cf}\mu)$ and $A(\kappa,\lambda',\mathrm{cf}\mu)$ hold.

Now let $\mu = \kappa$. If $\mathrm{cf}\lambda^\mu > \lambda^{<\mu}$ or $\mathrm{cf}\lambda^\mu \leq \kappa$ we are done. In the remaining case λ^μ is singular and we may apply Corollary 2.11. □

We illustrate the scope of Theorem 3.5 by an example. By Theorem 2.3, $A(\aleph_0, 2^{\aleph_0}, \aleph_0)$ and $D(\aleph_0, 2^{\aleph_0}, \aleph_0)$ hold, so the first interesting case occurs for $\kappa = \aleph_1$. The results in Section 2 allow us to conclude only that $A(\aleph_1, 2^{\aleph_1})$ and $D(\aleph_1, 2^{\aleph_1})$ hold when $2^{\aleph_0} = \aleph_1$ (using Theorem 2.3) or $2^{\aleph_0} = 2^{\aleph_1}$ (using Theorem 2.2(b) and the fact that $A(\aleph_0, 2^{\aleph_0})$ holds). If, however, $2^{\aleph_0} < \aleph_{\omega_1}$ and $2^{\aleph_1} > 2^{\aleph_0}$, then by Theorem 3.5, $A(\aleph_1, 2^{\aleph_1}, \aleph_1)$ holds, and if $\mathrm{cf}\, 2^{\aleph_1} > 2^{\aleph_0}$ then $D(\aleph_1, 2^{\aleph_1}, \aleph_1)$ holds. Putting these results together, we see that if either $2^{\aleph_0} = 2^{\aleph_1}$ or $2^{\aleph_0} < \aleph_{\omega_1}$, then $A(\aleph_1, 2^{\aleph_1})$ holds. The simplest case where $A(\aleph_1, 2^{\aleph_1})$ could fail is therefore

(1) $2^{\aleph_0} = \aleph_{\omega_1}$ and $2^{\aleph_1} = \aleph_{\omega_1+1}$.

The simplest cases where $D(\aleph_1, 2^{\aleph_1})$ could fail are (1) and

(2) $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_{\omega_2}$.

Mitchell [16] has shown that it is consistent with ZFC for either situation (1) or (2) to occur and $D(\aleph_1, 2^{\aleph_1})$ to be false. We will prove in Section 5 that it is consistent for situation (1) to occur and $A(\aleph_1, 2^{\aleph_1})$ to be false. This implies Mitchell's result for situation (1) by Theorem 2.2(d). Note that Mitchell's result for situation (2) shows that it is consistent for $A(\kappa,\lambda,\mu)$ to hold and $D(\kappa,\lambda,\mu)$ to fail when μ is regular. Of course by Theorem 3.4, if GCH is assumed then $A(\kappa, \lambda, \mu)$ and $D(\kappa,\lambda,\mu)$ are equivalent whenever μ is regular.

If one is interested only in the independence of $A(\aleph_1, 2^{\aleph_1}, \aleph_1)$, then a much simpler situation than (1) can be found, namely:

$$2^{\aleph_0} = 2^{\aleph_1} = \aleph_3 . \tag{3}$$

Note that $A(\aleph_1, \aleph_2, \aleph_1)$ holds by Theorem 2.8. Situation (3) will also be treated in Section 5.

Another natural question is whether $A(\kappa, 2^\kappa, \kappa, \nu)$ (or even $A(\kappa, \kappa^+, \kappa, \nu)$) is possible when $\nu < \kappa$. By Theorem 3.4 this is not possible assuming GCH. It turns out that $A(\kappa, \kappa^+, \kappa, \nu)$ can hold only under very unusual circumstances. Shelah proved [17, Lemma 3.2] that if $A(\kappa, 2^\kappa, \kappa, \aleph_0)$ holds then either $2^\kappa = 2^{\aleph_0}$ or $2^{|\alpha|} = 2^\kappa$ for some regular $\aleph_\alpha \leq \kappa$. The following is an improvement of that result. Except for the second sentence, the following theorem is implicitely contained in [17].

Theorem 3.7. *Let κ be the least cardinal such that $\nu \leq \kappa$ and $A(\kappa,\lambda,\kappa,\nu)$ holds, and assume $\mathrm{cf}\lambda > \kappa$. Then $\kappa = \nu$ or $\mathrm{cf}\kappa = \omega$. Moreover, if $\kappa \neq \nu$ then $\kappa = \aleph_\kappa$ and there is $\rho < \kappa$ such that $\rho^\nu \geq \lambda$.*

Proof. Assume $\kappa \neq \nu$, and let F be a $(\kappa,\lambda,\kappa,\nu)$-family. If $\mathrm{cf}\kappa > \omega$, then for each $X \in F$ there is an ordinal α_X such that $\nu < \alpha_X < \kappa$ and $|X \cap \alpha_X|$ $|\alpha_X|$. Since $\mathrm{cf}\lambda > \kappa$ there is α_0 such that $|\{X: \alpha_X = \alpha_0\}| = \lambda$. But then $\{X \cap \alpha_0: \alpha_X = \alpha_0\}$ is a $(|\alpha_0|, \lambda, |\alpha_0|, \nu)$-family of subsets of α_0, contradiction. Hence $\mathrm{cf}\kappa = \omega$.

Now let μ be a regular cardinal such that $\nu < \mu < \kappa$. Let ρ be the least cardinal such that $A(\rho,\lambda,\mu,\nu)$ holds. By Corollary 3.3, $\mathrm{cf}\rho = \mu$. Hence $\rho < \kappa$. But also $\rho \neq \mu$, so $\rho \geqslant \aleph_\mu$. It follows that if $\alpha < \kappa$ then $\aleph_\alpha < \kappa$, and hence $\kappa = \aleph_\kappa$. But we also have $\rho^\nu \geqslant \lambda$ by Theorem 2.7, and the proof is complete. □

In Section 6, we will show (as a special case of a more general theorem) that both $A(\aleph_1, \aleph_2, \aleph_1, \aleph_0)$ and its negation are consistent with $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.

4.

If D is an ultrafilter and κ is a cardinal, then κ^D is the cardinality of the ultrapower with respect to D of any structure of cardinality κ. All the facts we will need about ultrapowers may be found in [3].

Part (a) of the following is an observation of S. Shelah.

Theorem 4.1. *Let D be an ultrafilter on a cardinal ρ. Then*

(a) *$D(\kappa,\lambda)$ implies $D(\kappa^D, \lambda^D)$;*

(b) *If $\rho < \mu$, then $D(\kappa,\lambda,\mu)$ implies $D(\kappa^D, \lambda^D, \mu)$.*

Proof. (a) Let (S,U) be a (κ,λ)-pair, and suppose S is totally ordered by $<$. We may regard $(S, <, U)$ as a structure (with universe S, and with $<$ and U as binary and unary relations on S) appropriate for first-order logic. If $(S', <', U')$ is the ultrapower of $(S, <, U)$ with respect to D, then clearly (S', U') is a (κ^D, λ^D)-pair.

(b) Let (S,U) be a (κ, λ, μ)-pair, and let $F \subseteq (S - U) \times \mu \times U$ be such that for each $s \in S - U$, $\{(\alpha, u) \in \mu \times U: (s, \alpha, u) \in F\}$ is either a strictly increasing or strictly decreasing function with limit s. Let S be totally ordered by $<_1$, and let $<_2$ be the usual ordering on μ. Let

$$\mathfrak{A} = (S \cup \mu, S, \mu, U, <_1, <_2, F),$$

and let $(S' \cup M, S', M, U', <_1', <_2', F')$ be the ultrapower of $\mathfrak{A}$ with respect to D. We assert that the cofinality of M with respect to $<_2'$ is μ. Since

$\rho < \mu$ and μ is regular, it follows that if $f\colon \rho \to \mu$, then there is $\alpha < \mu$ with $f(\beta) < \alpha$ for all $\beta < \rho$. Hence, letting f_α be the constant function on ρ with value α, we see that the equivalence class of f is in the relation $<_2'$ to the equivalence class of f_α. Hence the equivalence classes of the functions f_α, $\alpha < \mu$, are cofinal in M, so M has cofinality μ. Now, using F', it follows immediately that $\chi(s) = \mu$ for all $s \in S' - U'$. □

If D is a regular ultrafilter on a cardinal ρ, it is well known (see [4]) that $\kappa^D = \kappa^\rho$ for all κ. Hence we have:

Corollary 4.2. (a) *$D(\kappa, \lambda)$ implies $D(\kappa^\rho, \lambda^\rho)$ for all ρ.*
(b) *$D(\kappa, \lambda, \mu)$ implies $D(\kappa^\rho, \lambda^\rho, \mu)$ for all ρ.*

Note that Corollary 4.2(b) is trivial for $\rho \geqslant \mu$, since then $\kappa^\rho = \lambda^\rho$. It is only when $\rho < \mu$ that we need apply Theorem 4.1.

We have not been able to obtain complete analogues of either Theorem 4.1 or Corollary 4.2 for property A. The difficulty seems to be that the almost-disjointness property is not as easily expressible in first-order logic as the dense-set property. Nevertheless we have some partial results:

Theorem 4.3. *Let $\nu \leqslant \mu \leqslant \kappa \leqslant \lambda$. Then $A(\kappa, \lambda, \mu, \nu)$ implies $A(\kappa^\rho, \lambda^\rho, \mu, \nu)$ for all ρ.*

Proof. Let F be a $(\kappa, \lambda, \mu, \nu)$-family. We may assume that each $X \in F$ has order-type μ (with respect to the usual ordering on κ). If $f\colon \rho \to F$, then we define a set X_f as follows. We let $g \in X_f$ iff $g\colon \rho \to \kappa$ and there is some $\alpha < \mu$ such that for all $\beta < \rho$, $g(\beta)$ is the α^{th} member of $f(\beta)$. It is easy to see that $\{X_f\colon f \in {}^F\rho\}$ is a $(\kappa^\rho, \lambda^\rho, \mu, \nu)$-family of subsets of ${}^\rho\kappa$.

A similar argument may be used to prove Corollary 4.2(b) directly.

While we have not been able to prove that $A(\kappa, \lambda)$ implies $A(\kappa^\rho, \lambda^\rho)$ generally, it is clear that this is true in many cases, for instance if $D(\kappa, \lambda)$ holds or if $A(\kappa, \lambda, \mu)$ holds for some μ. Part (b) of the next theorem gives still another condition.

We write $A'(\kappa, \lambda)$ to mean that there is a (κ, λ)-family F such that $|X|$ is regular for all $X \in F$. We do not know whether $A'(\kappa,\lambda)$ is equivalent to $A(\kappa,\lambda)$ for all κ and λ if GCH is not assumed. Under the GCH, of course, both notions are equivalent.

Theorem 4.4. *Let D be an ultrafilter on a cardinal ρ.*

(a) *If μ is regular and $\rho < \mu$, then $A(\kappa, \lambda, \mu)$ implies $A(\kappa^D, \lambda^D, \mu)$.*

(b) *$A'(\kappa, \lambda)$ implies $A'(\kappa^D, \lambda^D)$ (and hence $A'(\kappa, \lambda)$ implies $A'(\kappa^\rho, \lambda^\rho)$ for all ρ).*

Proof. We prove (a) and (b) simultaneously. Let F be a (κ, λ)-family. We may assume that

(1) for all $X \in F$, $|X|$ is regular and the order-type of X is $|X|$. It follows immediately that

(2) if $X, Y \in F$ and $X \neq Y$ then there is $\alpha \in X$ such that for all $\beta > \alpha$, either $\beta \notin X$ or $\beta \notin Y$.

Now let $\mathfrak{A} = (\kappa \cup F, \kappa, F, E, <)$, where E is the $\in$-relation restricted to $\kappa \times F$ and $<$ is the usual ordering on κ. Let $(K \cup F', K, F', E', <')$ be the ultrapower of $\mathfrak{A}$ with respect to D. For $f \in F'$, let $X_f = \{k \in K;\ k\ E'\ f\}$, and let $Y_f \subseteq X_f$ be a cofinal subset with order-type equal to the cofinality of X_f (with respect to $<'$). Then $|Y_f|$ is regular. Moreover, since (2) is expressible by a first-order sentence in $\mathfrak{A}$, we see that if $f \neq g$, then there are $k_1 \in Y_f$ and $k_2 \in Y_g$ such that if $k > k_1$ or $k > k_2$ then $k \notin Y_f \cap Y_g$. Hence Y_f and Y_g are almost-disjoint, so $\{Y_f: f \in F'\}$ is a (κ^D, λ^D)-family of subsets of K, and $A'(\kappa^D, \lambda^D)$ holds. This proves (b). If in addition $|X| = \mu$ for all $X \in F$ and $\rho < \mu$, then we may see $|Y_f| = \mu$ for all $f \in F'$ by an argument similar to the proof of Theorem 4.1(b). This proves (a). □

By combining the methods of this section with those of the preceding one, we obtain the following theorem, due to Shelah.

Theorem 4.5. *If $\aleph_\alpha^\lambda = \aleph_\alpha$ and $2^{\aleph_\alpha} \geqslant \aleph_{\alpha+\lambda^+}$, then $D(\aleph_\alpha, \aleph_{\alpha+\lambda^+})$ holds, and hence $A(\aleph_\alpha, \aleph_{\alpha+\lambda^+})$ holds.*

Proof. We prove by induction on $\beta \leqslant \lambda^+$ that $D(\aleph_\alpha, \aleph_{\alpha+\beta})$ holds. If β is a limit ordinal, this follows from Theorem 2.2(c). (Note that $\lambda < \aleph_\alpha$ since $\aleph_\alpha^\lambda = \aleph_\alpha$). Assume $\beta = \gamma + 1$, and let μ be the least cardinal such that $\aleph_\alpha^\mu \geqslant \aleph_{\alpha+\beta}$.

Case 1: $\mathrm{cf}\mu > \lambda$. We know $D(\aleph_{\alpha+\gamma}, \aleph_\alpha^\mu, \mathrm{cf}\mu)$ holds by Theorem 2.3. Let κ be the least cardinal such that $D(\kappa, \aleph_{\alpha+\beta}, \mathrm{cf}\mu)$ holds. By Corollary 3.2, $\mathrm{cf}\kappa = \mathrm{cf}\mu$. Since $\mathrm{cf}\mu > \lambda$ and $\mu \leqslant \aleph_\alpha$, this means $\kappa \leqslant \aleph_\alpha$.

Case 2: $\mathrm{cf}\mu \leqslant \lambda$. By inductive hypothesis, $D(\aleph_\alpha, \aleph_\alpha^{<\mu})$ holds. It is easy to see that $\aleph_\alpha^\mu = (\aleph_\alpha^{<\mu})^{\mathrm{cf}\mu}$. By Corollary 4.2(a), $D(\aleph_\alpha^\lambda, (\aleph_\alpha^{<\mu})^\lambda)$ holds. Since $\mathrm{cf}\mu \leqslant \lambda$, this means $D(\aleph_\alpha, \aleph_\alpha^\mu)$ holds. □

5.

In this section we apply a combinatorial theorem of Erdös and Rado to obtain independence results regarding properties A and D. For terminology concerning forcing and generic sets, see Section 1. We assume the reader is familiar with the approach to forcing taken, for example, in [19] or [10].

All the independence results in the remaining sections will be given in model-theoretic form. A typical theorem will read: If $\mathfrak{M}$ is a countable transitive model of ZFC (or ZFC + GCH) then in the Cohen extension $\mathfrak{M}$ [G], (*) is true. If (*) can be expressed without reference to parameters in $\mathfrak{M}$, then, in the customary manner (see [12, pp. 132–133]), the proof of the model-theoretic theorem can be converted into a proof of the theorem that if ZF is consistent then so is ZF + (*). We leave the details of such conversions to the reader.

The following is a special case of [8, Theorem 39].

Theorem 5.1 (Erdös-Rado). *For all cardinals* κ, $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$.

Unless explicitly stated to the contrary, for the remainder of this section $\mathfrak{M}$ is assumed to be a countable transitive model of ZFC, P is a partial ordering lying in $\mathfrak{M}$, and G is P-generic over $\mathfrak{M}$.

We shall need the following two well-known facts about forcing. Lemma 5.2 is easily proved by the method of Lemma 56 of [10]. Lemma 5.3 is the same as [10, Lemma 57].

Lemma 5.2. *The following is true in* $\mathfrak{M}$. *Assume P has the μ-chain condition, $p \in P$, τ is a term of the language of forcing and*

$$p \Vdash \tau \subset \mathfrak{M} \wedge |\tau| = \kappa.$$

Then there is a set $x \in \mathfrak{M}$ such that $p \Vdash \tau \subseteq x$ and if $\mu \leqslant \kappa$, then $|x| = \kappa$, while if $\kappa < \mu$, then $|x| < \mu$. Hence if $\nu \geqslant \mu$ and ν is regular in $\mathfrak{M}$, then ν remains regular in $\mathfrak{M}$ [G].

Lemma 5.3. *Assume that P is μ-complete in $\mathfrak{M}$. If $f \in \mathfrak{M}$ [G], $\alpha < \mu$, and $f\colon \alpha \to \mathfrak{M}$, then $f \in \mathfrak{M}$. Hence if $\kappa < \mu$ and κ is a cardinal in $\mathfrak{M}$, then κ remains a cardinal in $\mathfrak{M}$ [G] and its cofinality is unchanged.*

Note that as long as cofinalities are preserved (i.e. $(\mathrm{cf}\alpha)^{\mathfrak{M}} = (\mathrm{cf}\alpha)^{\mathfrak{M}[G]}$ for every ordinal $\alpha \in \mathfrak{M}$) in the passage from $\mathfrak{M}$ to $\mathfrak{M}[G]$, every cardinal of $\mathfrak{M}$ remains a cardinal in $\mathfrak{M}[G]$.

The main results of this section are derived from the next theorem.

Theorem 5.4. *Assume that in $\mathfrak{M}$, P has the ν-chain condition and is ρ-complete. Assume also that cofinalities are preserved in $\mathfrak{M}[G]$. Let κ and μ be cardinals in $\mathfrak{M}[G]$. Then, in $\mathfrak{M}[G]$,*

(a) *if $\mu < \rho$ or $\mathrm{cf}\mu \geqslant \nu$, then $A(\kappa,\lambda^+,\mu)$ is false, where $\lambda = (\kappa^\mu)^{\mathfrak{M}}$.*

(b) *if $\mu > \nu$ but $\mathrm{cf}\mu < \rho$, then $A(\kappa,\lambda^+,\mu)$ is false, where*

$$\lambda = \max\left((\kappa^\mu)^{\mathfrak{M}}, (2^{(\mu^{\mathrm{cf}\mu})})^{\mathfrak{M}}\right).$$

Proof. (a) If $\mu < \rho$, then by Lemma 5.3, $(\kappa^\mu)^{\mathfrak{M}} = (\kappa^\mu)^{\mathfrak{M}[G]}$. Since $A(\kappa,\lambda',\mu)$ implies $\lambda' \leqslant \kappa^\mu$ by Theorem 2.7, we are done immediately in this case. Now assume $\mathrm{cf}\mu \geqslant \nu$. We work in $\mathfrak{M}[G]$. Let F be a (κ,λ^+,μ)-family. By Lemma 5.2, for each $X \in F$ there is $X' \in \mathfrak{M}$ such that $|X'| = \mu$ and $X \subseteq X'$. Since $\lambda^+ > (\kappa^\mu)^{\mathfrak{M}}$ there is a single $X'_0 \in \mathfrak{M}$ such that $|\{X \in F\colon X \subseteq X'_0\}| = \lambda$. Then $\{X \in F\colon X \subseteq X'_0\}$ is a (μ,λ^+,μ)-family, so in order to show that $A(\kappa,\lambda^+,\mu)$ is false we need only show that $A(\mu,\lambda^+,\mu)$ is false. There are two cases:

Case 1: μ is regular. Let F be a (μ,λ^+,μ)-family in $\mathfrak{M}[G]$. Say $F = \{X_\alpha\colon \alpha < \lambda^+\}$. Let $H = \{(\alpha,\beta)\colon \alpha < \lambda^+ \text{ and } \beta \in X_\alpha\}$. Then in $\mathfrak{M}[G]$, H satisfies

(1) $H \subseteq \lambda^+ \times \mu$ and for all $\alpha < \lambda^+$, $|\{\beta\colon (\alpha,\beta) \in H\}| = \mu$, and if $\alpha,\alpha' < \lambda^+$ and $\alpha \neq \alpha'$, then $|\{\beta\colon (\alpha,\beta), (\alpha',\beta) \in H\}| < \mu$.

Let $\dot{H}$ be a term of the language of forcing which denotes H in $\mathfrak{M}[G]$, and let $p \in G$ be such that p forces (1) with H replaced by $\dot{H}$. We assume that the assertion that cofinalities are preserved is forced by p also.

Now we work in $\mathfrak{M}$. For $\alpha,\alpha' < \lambda^+$, let

$$Z_{\alpha\alpha'} = \{\gamma < \mu\colon (\exists q \leqslant p) q \Vdash \sup\{\beta\colon (\alpha,\beta), (\alpha',\beta) \in \dot{H}\} = \gamma\}.$$

Since P has the ν-chain condition, we have $|Z_{\alpha\alpha'}| < \nu$. Since μ is regular we have $\gamma_{\alpha\alpha'} = \sup Z_{\alpha\alpha'} < \mu$. By Theorem 5.1, $(2^\mu)^+ \to (\mu^+)^2_\mu$, and since $\lambda^+ \geqslant (2^\mu)^+$ we conclude that there are $Y \subseteq \lambda^+$ and $\gamma < \mu$ such that $|Y| = \mu^+$ and $\gamma_{\alpha\alpha'} = \gamma$ for all $\alpha,\alpha' \in Y$ with $\alpha \neq \alpha'$. Clearly $p \Vdash \{\beta\colon (\alpha,\beta), (\alpha'\beta) \in \dot{H}\} \subseteq \gamma$ for distinct $\alpha,\alpha' \in Y$.

Now we return to $\mathfrak{M}[G]$. By the remark just made we must have $(X_\alpha - \gamma) \cap (X_{\alpha'} - \gamma) = 0$ whenever $\alpha,\alpha' \in Y$ and $\alpha \neq \alpha'$. Of course $|X_\alpha - \gamma| = \mu$ for all α. Since $|Y| = \mu^+$ in $\mathfrak{M}[G]$ as well as in $\mathfrak{M}$, it follows that $\{X_\alpha - \gamma : \alpha \in Y\}$ is a collection of μ^+ pairwise disjoint non-empty subsets of μ, a contradiction.

Case 2: μ is singular. Let F, H, $\dot{H}$ and p be as in Case 1. Let $\alpha < \lambda^+$. We work in $\mathcal{M}$. By Lemma 5.2, for every cardinal $\mu' < \mu$ with $\mu' \geqslant \nu$ there is a set $x_{\alpha\mu'} \subseteq \mu$ such that $|x_{\alpha\mu'}| = \mu'$ and

$$p \Vdash \text{the first } \mu' \text{ elements of } \{\beta : (\alpha,\beta) \in \dot{H}\} \text{ lie in } x_{\alpha\mu'} .$$

Since there are at most 2^μ such sequences $\langle x_{\alpha\mu'} : \nu \leqslant \mu' < \mu, \mu' \text{ a cardinal}\rangle$ and $2^\mu \leqslant \lambda$, it is clear that there is a sequence $\langle x_{\mu'} : \nu < \mu' < \mu\rangle$ such that

$$|\{\alpha < \lambda^+ : \text{for all } \mu', x_{\alpha\mu'} = x_{\mu'}\}| = \lambda^+ .$$

It follows immediately that we may assume that in $\mathcal{M}[G]$

(2) for all $\alpha < \lambda^+$ and all cardinals μ' with $\nu \leqslant \mu' < \mu$, $|\{\beta < \mu' : (\alpha,\beta) \in H\}| = \mu'$.

We assume also that p forces (2) with H replaced by $\dot{H}$.

Now let $\alpha,\alpha' < \lambda^+$ with $\alpha \neq \alpha'$. We return to $\mathcal{M}$. Let

$$W_{\alpha\alpha'} = \{\mu' < \mu : (\exists q \leqslant p) q \Vdash |\{\beta : (\alpha,\beta), (\alpha',\beta) \in \dot{H}\}| = \mu'\} .$$

Since P has the ν-chain condition, $|W_{\alpha\alpha'}| < \nu \leqslant \mathrm{cf}\mu$, so $\mu_{\alpha\alpha'} = \max(\nu, \sup W_{\alpha\alpha'}) < \mu$. Then

$$p \Vdash |\{\beta : (\alpha,\beta), (\alpha',\beta) \in \dot{H}\}| \leqslant \mu_{\alpha\alpha'} .$$

Just as in Case 1, we may now find $\delta_{\alpha\alpha'} < \mu^+_{\alpha\alpha'}$ such that $\mu_{\alpha\alpha'} \leqslant \delta_{\alpha\alpha'}$ and

$$p \Vdash \{\beta : \beta < \mu^+_{\alpha\alpha'}, (\alpha,\beta), (\alpha',\beta) \in \dot{H}\} \subseteq \delta_{\alpha\alpha'} .$$

Since by Theorem 5.1, $\lambda^+ \to (\mu^+)^2_\mu$, there are $W \subseteq \lambda^+$ and $\delta < \mu$ such that $|W| = \mu^+$ and $\delta_{\alpha\alpha'} = \delta$ (and $\mu_{\alpha\alpha'} = |\delta|$) for all $\alpha,\alpha' \in W$ with $\alpha \neq \alpha'$. But then, as in Case 1, we reach a contradiction by observing that in $\mathcal{M}[G]$, $\langle\{\beta : \beta \in X_\alpha \text{ and } \delta \leqslant \beta < |\delta|^+\} : \alpha \in W\rangle$ is a sequence of μ^+ pairwise disjoint non-empty subsets of $|\delta|^+$.

(b) As in part (a), it will suffice to show that $A(\mu,\lambda^+,\mu)$ is false. Let F be a (μ,λ^+,μ)-family in $\mathcal{M}[G]$. Say $F = \{X_\alpha : \alpha < \lambda^+\}$. Arguing as in Case 2, we may assume that if $\nu \leqslant \mu' < \mu$ then $|X_\alpha \cap \mu'| = \mu'$, i.e. that (2) holds when H is defined from F as in Case 2. Let $\dot{H}$ and p be as in Case 2 also. We work in $\mathcal{M}$. Let μ_α, $\alpha < \mathrm{cf}\mu$, be an increasing sequence of cardinals with limit μ such that $\mu_0 \geqslant \nu$. Let $\alpha,\alpha' < \lambda^+$ with $\alpha \neq \alpha'$. Arguing as in Case 1, for each $\beta < \mathrm{cf}\mu$ there is $\delta_{\alpha\alpha'\beta} < \mu^+_\beta$ such that

(3) $p \Vdash$ if $|\{\gamma < \mu^+_\beta : (\alpha,\gamma), (\alpha',\gamma) \in \dot{H}\}| < \mu^+_\beta$,

then $\{\gamma < \mu^+_\beta : (\alpha,\gamma), (\alpha',\gamma) \in \dot{H}\} \subseteq \delta_{\alpha\alpha'\beta}$.

There are at most $\mu^{\mathrm{cf}\mu}$ such sequences $\langle\delta_{\alpha\alpha'\beta}\colon \beta<\mathrm{cf}\mu\rangle$, so by Theorem 5.1 (and noting that $\lambda^+ \geqslant (2^{(\mu^{\mathrm{cf}\mu})})^+$) we obtain $V\subseteq\lambda^+$ and $\langle\delta_\beta\colon \beta<\mathrm{cf}\mu\rangle$ such that $|V| = (\mu^{\mathrm{cf}\mu})^+$ and $\delta_{\alpha\alpha'\beta} = \delta_\beta$ for all $\alpha,\alpha'\in V$ with $\alpha\neq\alpha'$ and all $\beta<\mathrm{cf}\mu$. For each $\alpha\in V$ and $\beta<\mathrm{cf}\mu$, choose $\gamma_{\alpha\beta}\in\{\gamma\in X_\alpha\colon \delta_\beta<\gamma<\mu_\beta^+\}$. Note that if $\alpha\neq\alpha'$ and $\gamma_{\alpha\beta}=\gamma_{\alpha'\beta}$, then $|\{\gamma\in\mu_\beta^+\colon (\alpha,\gamma),(\alpha',\gamma)\in H\}| = \mu_\beta^+$ by (3). By Lemma 5.3 we have $(\mu^{\mathrm{cf}\mu})^{\mathfrak{M}} = (\mu^{\mathrm{cf}\mu})^{\mathfrak{M}[G]}$. Hence there exist $\alpha,\alpha'\in V$ such that $\alpha\neq\alpha'$ and $\gamma_{\alpha\beta}=\gamma_{\alpha'\beta}$ for all $\beta<\mathrm{cf}\mu$. But then $|X_\alpha\cap X_{\alpha'}|=\mu$, a contradiction. This completes the proof of Theorem 5.4. □

It seems very likely that in Theorem 5.4(b) we can take $\lambda=(\kappa^\mu)^{\mathfrak{M}}$, but we have not been able to prove this stronger statement.

Theorem 5.4 is best understood by considering some particular partial orderings. If ν and ρ are cardinals and ν is regular, then let $P(\nu,\rho)$ be the set of all functions f such that domain $f\subseteq\rho$, range $f\subseteq 2$, and $|\text{domain } f| < \nu$. If $p,q\in P(\nu,\rho)$, then we let $p\leqslant q$ iff $p\supseteq q$. $P(\nu,\rho)$ is a standard partial ordering used for adding ρ new subsets of ν to $\mathfrak{M}$.

The following fact is well-known (see [10, pp. 69–70]).

Lemma 5.5. (GCH) *$P(\nu,\rho)$ has the ν^+-chain condition and is ν-complete.*

Theorem 5.6. *Assume that $\mathfrak{M}$ is also a model of* GCH, *and that $P = P(\nu,\rho)^{\mathfrak{M}}$. Then cofinalities are preserved in $\mathfrak{M}[G]$. Let κ and μ be cardinals in $\mathfrak{M}[G]$ with $\mu\leqslant\kappa$, and consider the following situations:*

(4) $\kappa<\nu$ *or* $\kappa>\rho$

(5) $\nu\leqslant\kappa<\mathrm{cf}\rho$

(6) $\max(\nu,\mathrm{cf}\rho)\leqslant\kappa\leqslant\rho$.

Then, in $\mathfrak{M}[G]$,

(a)
$$2^\kappa = \begin{cases}\kappa^+ & \textit{if (4) holds},\\ \rho & \textit{if (5) holds},\\ \rho^+ & \textit{if (6) holds};\end{cases}$$

(b) *if either* (4) *holds or* $\mathrm{cf}\mu>\nu$ *or* $\mu<\nu$, *then $A(\kappa,\kappa^+,\mu)$ is false unless* $\mathrm{cf}\mu=\mathrm{cf}\kappa$, *in which case $A(\kappa,\kappa^{++},\mu)$ is false;*

(c) *if* $\mathrm{cf}\mu=\nu$ *and* (5) *or* (6) *holds, then $D(\kappa,2^\nu,\nu)$ and $A(\kappa,2^\nu,\mu)$ hold;*

(d) *if* $\mathrm{cf}\mu<\nu$ *and* $\mu>\nu$ *and* (5) *or* (6) *holds, then $A(\kappa,\kappa^+,\mu)$ is false unless* $\mathrm{cf}\mu=\mathrm{cf}\kappa$ *or* $\mu\leqslant\kappa\leqslant\mu^+$. *If* $\mu\leqslant\kappa\leqslant\mu^+$, *then $A(\kappa,\mu^{+++},\mu)$ is false. If* $\mathrm{cf}\kappa=\mathrm{cf}\mu$ *and* $\kappa>\mu$, *then $A(\kappa,\kappa^{++},\mu)$ is false.*

Proof. The fact that cofinalities are preserved is a well-known consequence of Lemmas 5.2, 5.3 and 5.5. The following table for cardinal exponentiation is easily computed using the methods in [10]. If $\mu \leqslant \kappa$ then

$$\kappa^{\mu} = \begin{cases} \kappa & \text{if } \mu < \mathrm{cf}\kappa \text{ and (4) holds,} \\ \kappa^{+} & \text{if } \mu \geqslant \mathrm{cf}\kappa \text{ and (4) holds,} \\ \max(\rho,\kappa) & \text{if } \mu < \mathrm{cf}\kappa \text{ and (5) holds,} \\ \max(\rho,\kappa^{+}) & \text{if } \mu \geqslant \mathrm{cf}\kappa \text{ and (5) holds,} \\ \max(\rho^{+},\kappa) & \text{if } \mu < \mathrm{cf}\kappa \text{ and (6) holds.} \\ \max(\rho^{+},\kappa^{+}) & \text{if } \mu \geqslant \mathrm{cf}\kappa \text{ and (6) holds.} \end{cases}$$

Of course, this proves (a).

(b) Let κ' be the least cardinal such that $A(\kappa',\kappa^{+},\mu)$ holds. If $\kappa' \leqslant \kappa$ and $\mathrm{cf}\kappa \neq \mathrm{cf}\mu$ then by Corollary 3.3, $\kappa' < \kappa$ and $\mathrm{cf}\kappa' = \mathrm{cf}\mu$. If (4) holds then $2^{\kappa'} < \kappa^{+}$, a contradiction. If $\mu < \nu$ or $\mathrm{cf}\mu > \nu$, then $(\kappa'^{\mu})^{\mathfrak{M}} \leqslant \kappa$ so $A(\kappa',\kappa^{+},\mu)$ is false by Theorem 5.4(a). Now assume that $\mathrm{cf}\mu = \mathrm{cf}\kappa$. If (4) holds then $\kappa^{\mu} = \kappa^{+}$, so $A(\kappa,\kappa^{++},\mu)$ is false. If $\mu < \nu$ or $\mathrm{cf}\mu > \nu$, then, since $(\kappa^{\mu})^{\mathfrak{M}} = \kappa^{+}$, we apply Theorem 5.4(a) to conclude that $A(\kappa,\kappa^{++},\mu)$ is false.

(c) Since $2^{<\nu} = \nu$, it follows from Theorem 2.3 that $D(\nu,2^{\nu},\nu)$ holds, so $D(\kappa,2^{\nu},\nu)$ holds for all $\kappa \geqslant \nu$. Hence $A(\kappa,2^{\nu},\nu)$ holds, so $A(\kappa,2^{\nu},\mu)$ holds by Theorem 2.9.

(d) Let κ' be the least cardinal such that $A(\kappa',\kappa^{+},\mu)$ holds. If $\kappa' \leqslant \kappa$ and $\mathrm{cf}\kappa \neq \mathrm{cf}\mu$ then $\kappa' < \kappa$ and $\mathrm{cf}\kappa' = \mathrm{cf}\mu$ by Corollary 3.3. If either $\kappa' \neq \mu$ or $\kappa'^{+} < \kappa$, then $A(\kappa',\kappa^{+},\mu)$ is false by Theorem 5.4(b), a contradiction. This proves the first assertion of (d). The other assertions follow from Theorem 5.4(b) also. □

It seems very likely that in Theorem 5.6(d) we can replace μ^{+++} by μ^{++}. This corresponds to the case left open in Theorem 5.4.

To be quite specific, let $\nu = \aleph_0$ and $\rho = \aleph_3$ in Theorem 5.6. Then $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$ in $\mathfrak{M}[G]$, and $A(\aleph_0,\aleph_3,\aleph_0)$ holds, while $A(\aleph_1,\aleph_3,\aleph_1)$ (and hence $D(\aleph_1,\aleph_3,\aleph_1)$) is false.

If $\nu = \aleph_0$ and $\rho = \aleph_{\omega_1}$, then in $\mathfrak{M}[G]$, $2^{\aleph_0} = \aleph_{\omega_1}$ and $2^{\aleph_1} = \aleph_{\omega_1+1}$. Both $A(\aleph_1,\aleph_{\omega_1+1},\aleph_0)$ and $A(\aleph_1,\aleph_3,\aleph_1)$ are false, so $A(\aleph_1,2^{\aleph_1})$ is false, and hence $D(\aleph_1,2^{\aleph_1})$ is false also. These are the results promised in Section 3.

The reader familiar with Easton's method [5] for forcing with many of the partial orderings $P(\nu,\rho)$ simultaneously will see that it can be applied to Theorem 5.4. Since this application does not require any ideas not already presented, we offer the following theorem without proof. For simplicity, we consider only the properties $A(\kappa,\lambda,\kappa)$ and $D(\kappa,\lambda,\kappa)$ when κ is regular.

Theorem 5.7. *Assume that $\mathfrak{M}$ is a countable transitive model of Gödel-Bernays set theory +* GCH. *Let X and Y be disjoint classes of $\mathfrak{M}$ such that, in $\mathfrak{M}$,*

(7) *$X \cup Y$ is the class of all regular cardinals,*

(8) *$\aleph_0 \in X$,*

(9) *if κ is singular and $X \cap \kappa$ is cofinal in κ, then $\kappa^+ \notin X$,*

(10) *if κ is inaccessible but not Mahlo, and if $X \cap \kappa$ is cofinal in κ, then $\kappa \in X$.*

Let F be a function in $\mathfrak{M}$ mapping X into the class of all cardinals of $\mathfrak{M}$ with the properties

(11) *if $\kappa, \lambda \in X$ and $\kappa < \lambda$, then $F(\kappa) \leqslant F(\lambda)$,*

(12) *if $\kappa \in X$, then $\mathrm{cf}\,(F(\kappa)) > \kappa$.*

Then there is a Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$ such that in $\mathfrak{N}$ cofinalities are preserved, and in addition:

(13) *if $\kappa \in X$, then $2^\kappa = F(\kappa)$,*

(14) *if $\kappa \in Y$ and $\lambda = \sup\{F(\mu) \colon \mu \in X$ and $\mu < \kappa\}$, then*

$$2^\kappa = \begin{cases} \lambda & \text{if } \kappa < \mathrm{cf}\lambda\,, \\ \lambda^+ & \text{if } \mathrm{cf}\lambda \leqslant \kappa < \lambda\,, \\ \kappa^+ & \text{if } \lambda \leqslant \kappa\,, \end{cases}$$

(15) *if $\kappa \in X$, then $D(\kappa, 2^\kappa, \kappa)$ holds,*

(16) *if $\kappa \in Y$, then $A(\kappa, \kappa^{++}, \kappa)$ is false.*

For the record, the partial ordering P used to obtain $\mathfrak{N}$ is the set of all functions f with domain $f \subseteq X$ (domain f is a *set*) and such that

(17) for all $\kappa \in$ domain f, $f(\kappa) \in P(\kappa, F(\kappa))$,

(18) if λ is regular, then $|\lambda \cap \text{domain } f| < \lambda$.

We put $f \leqslant g$ iff domain $f \supseteq$ domain g and $f(\kappa) \leqslant g(\kappa)$ for all $\kappa \in$ domain g. Of course, if X is a proper class then so is P.

Conditions (9) and (10) were overlooked in [2, Theorem 3.2.5]. Roughly speaking, the reason why (10) is necessary is that if κ is inaccessible but not Mahlo, $X \cap \kappa$ is cofinal in κ and $\kappa \notin X$, then that part of the Easton partial ordering P which is used to deal with $F|\kappa$ (namely $\{f|\kappa \colon f \in P\}$) has the κ^+-chain condition but not the κ-chain condition, so the method of Theorem 5.4 cannot be applied to κ to deduce that $A(\kappa, \kappa^{++}, \kappa)$ is false in $\mathfrak{N}$. In fact, it is not difficult to see that if $\kappa \notin X$, then in $\mathfrak{N}$ we would still have $D(\kappa, 2^\kappa, \kappa)$ true.

(9) is necessary for much the same reason. Under the hypotheses of (9), that part of P used to deal with $F|\kappa$ has the κ^{++}-chain condition but not the κ^+-chain condition.

The methods of this section do not seem to yield a Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$ in which

(19) $2^{\aleph_0} = \aleph_{\omega_1}$, $2^{\aleph_1} > \aleph_{\omega_1+1}$ and $A(\aleph_1, 2^{\aleph_1})$ is false.

Mitchell [16] remarks that if there is a model $\mathfrak{M}$ in which $2^{\aleph_1} < \aleph_{\omega_1}$ and $\aleph_{\omega_1}^{\aleph_1} > \aleph_{\omega_1+1}$, then, forcing via $P(\aleph_0, \aleph_{\omega_1})^{\mathfrak{M}}$, we would obtain the consistency of (19) with A replaced by D. We add the remark that, in the same situation, the consistency of (19) could be obtained by the methods of this section.

6.

This section is concerned with the propositions $A(\kappa,\lambda,\kappa,\nu)$ when $\nu < \kappa$. If the GCH is assumed, then by Theorem 3.4(a) these propositions are never true. Nevertheless, it will follow from the results of this section that, for example, both $A(\aleph_1, \aleph_2, \aleph_1, \aleph_0)$ and its negation are consistent with ZFC + $2^{\aleph_0} \geqslant \aleph_2$.

The partial ordering for proving $A(\aleph_1, \aleph_2, \aleph_1, \aleph_0)$ to be consistent is quite simple. By Theorem 2.8 there is an $(\aleph_1, \aleph_2, \aleph_1)$-family F. Let $\langle F_\alpha : \alpha < \omega_2 \rangle$ enumerate F. Let Q be the set of all functions f such that domain f is a finite subset of ω_2 and for all $\alpha \in$ domain f, $f(\alpha)$ is a finite subset of F_α. We let $f \leqslant g$ iff

$$\text{domain } g \subseteq \text{domain } f\,,$$

$$g(\alpha) \subseteq f(\alpha) \quad \text{for all } \alpha \in \text{domain } g\,,$$

$$f(\alpha) \cap f(\beta) = g(\alpha) \cap g(\beta) \quad \text{for all } \alpha,\beta \in \text{domain } g \text{ with } \alpha \neq \beta\,.$$

It is not difficult to see that Q has the countable chain condition and hence preserves cofinalities (and cardinals). Furthermore, if G is Q-generic over a model $\mathfrak{M}$, then $\{G_\alpha : \alpha < \omega_2\}$ is an $(\aleph_1, \aleph_2, \aleph_1, \aleph_0)$-family in $\mathfrak{M}[G]$, where $G_\alpha = \bigcup \{f(\alpha) : f \in G\}$ for each $\alpha < \omega_2$. It is natural to say that the G_α are obtained by *thinning out* the F_α.

The general construction is a little more elaborate. To get the consistency of $A(\aleph_2, \aleph_3, \aleph_2, \aleph_0)$ we must first thin out an $(\aleph_2, \aleph_3, \aleph_2)$-family to an $(\aleph_2, \aleph_3, \aleph_2, \aleph_1)$-family and then thin that out to an $(\aleph_2, \aleph_3, \aleph_2, \aleph_0)$ family. In general, this thinning-out procedure must be iterated many times.

Let κ, λ and ν be cardinals such that ν is regualr and $\nu \leqslant \kappa \leqslant \lambda$, and let $F = \langle F_\alpha : \alpha < \lambda \rangle$ be a sequence of subsets of κ, possibly with repetitions. Then $Q'(\kappa, \lambda, \nu, F)$ is the set of all functions f such that

(1) domain $f \subseteq \lambda$ and $|\text{domain } f| < \nu$, and

(2) for all $\alpha \in$ domain f, $f(\alpha) \subseteq F_\alpha$ and $|f(\alpha)| < \nu$.

Let $f \leqslant g$ iff

(3) domain $g \subseteq$ domain f

(4) $g(\alpha) \subseteq f(\alpha)$ for all $\alpha \in$ domain g, and

(5) $g(\alpha) \cap g(\beta) = f(\alpha) \cap f(\beta)$ for all $\alpha, \beta \in$ domain g with $\alpha \neq \beta$.

Now let K be the set of all regular cardinals μ such that $\nu \leqslant \mu \leqslant \kappa$. The set $Q(\kappa,\lambda,\nu,F)$ consists of all functions $f = \langle f_\mu : \mu \in K \rangle \in \Pi_{\mu \in K}\, Q'(\kappa,\lambda,\mu,F)$ satisfying

(6) if $\mu, \mu' \in K$ and $\mu < \mu'$, then domain $f_\mu \subseteq$ domain $f_{\mu'}$ and $f_\mu(\alpha) \subseteq f_{\mu'}(\alpha)$ for all $\alpha \in$ domain f_μ.

We put $f \leqslant g$ iff $f_\mu \leqslant g_\mu$ for all $\mu \in K$. If $F_\alpha = \kappa$ for all α, then we write $Q(\kappa,\lambda,\nu)$ instead of $Q(\kappa,\lambda,\nu,F)$.

Throughout the rest of this section, $\mathcal{M}$ is a countable transitive model of ZFC + GCH.

Theorem 6.1. *Assume that in $\mathcal{M}$, κ, λ and ν are cardinals such that ν is regular and $\nu \leqslant \kappa \leqslant \lambda$. If G is $Q(\kappa,\lambda,\nu)^{\mathcal{M}}$-generic over $\mathcal{M}$, then in $\mathcal{M}[G]$ cofinalities (and hence cardinals) are preserved and $A(\kappa,\lambda,\kappa,\nu)$ is true.*

Proof. For each $\alpha < \lambda$, let $G_\alpha = \bigcup\{f_\nu(\alpha) : f \in G\}$. It is clear that if cofinalities are preserved in $\mathcal{M}[G]$, then $\{G_\alpha : \alpha < \lambda\}$ is a $(\kappa,\lambda,\kappa,\nu)$-family. The proof that cofinalities are preserved is broken up into a series of lemmas.

Lemma 6.2 is due to Erdos and Rado [7]. A simple proof may be found in [15]. A family F of sets is a *Δ-system* iff $A \cap B = \bigcap F$ for all $A, B \in F$ with $A \neq B$.

Lemma 6.2. *Let κ, λ be cardinals with $\kappa \geqslant \aleph_0$, and assume $\kappa^\lambda = \kappa$. Then for any family F of sets such that $|F| = \kappa^+$ and $|A| \leqslant \lambda$ for all $A \in F$ there is a Δ-system $F' \subseteq F$ with $|F'| = \kappa^+$.*

Let K be as in the definition of $Q(\kappa,\lambda,\nu,F)$. For $\mu \in K$, let $Q_\mu(\kappa,\lambda,\nu,F) = \{f | \{\mu' \in K : \mu' \leqslant \mu\} : f \in Q(\kappa,\lambda,\nu,F)\}$.

Lemma 6.3. *If $\mu^{<\mu} = \mu$ and $|F_\alpha \cap F_\beta| \leqslant \mu$ for $\alpha < \beta < \lambda$, then $Q_\mu(\kappa,\lambda,\nu,F)$ has the μ^+-chain condition. Hence if $\kappa^{<\kappa} = \kappa$ then $Q(\kappa,\lambda,\nu)$ has the κ^+-chain condition. In any case, $Q(\kappa,\lambda,\nu)$ has the $(2^\kappa)^+$-chain condition.*

Proof. Suppose on the contrary that there is a set I of pairwise incompatible elements with $|I| = \mu^+$. We may assume that for some $\mu' < \mu$ we have $|\text{domain } f_\mu| = \mu'$ for all $f \in I$. By Lemma 6.2 and the fact that $\mu^{\mu'} =$

μ, we may also assume that there is a set D such that

$$\operatorname{domain} f_\mu \cap \operatorname{domain} g_\mu = D \quad \text{for all } f, g \in I, \quad f \neq g.$$

Another application of Lemma 6.2 allows us to assume that there is a set R such that

$$\bigcup \operatorname{range}(f_\mu|D) \cap \bigcup \operatorname{range}(g_\mu|D) = R \quad \text{for all } f, g \in I, \quad f \neq g\,.$$

Let

$$X = \bigcup\{F_\alpha \cap F_\beta \colon \alpha, \beta \in D, \quad \alpha \neq \beta\}\,.$$

Since $|D| < \mu$ and $|F_\alpha \cap F_\beta| \leqslant \mu$ for $\alpha \neq \beta$, we have $|X| \leqslant \mu$. Let $Y = X \cup R$.

Now we assert that there is $f \in Q_\mu(\kappa,\lambda,\nu,F)$ such that

$$I' = \{g \in I\colon \text{ for all } \tau \in \operatorname{domain} g \text{ and all } \alpha \in D \cap \operatorname{domain} g_\tau,$$
$$g_\tau(\alpha) \cap Y = f_\tau(\alpha)\}$$

has power μ^+. This is clear if μ is a successor cardinal. Assume μ is a limit cardinal. Since μ is regular, it is easy to see that for any $g \in Q_\mu(\kappa,\lambda,\nu,F)$, there is $\sigma < \mu$ such that $g_\sigma = g_\tau$ for all τ with $\sigma \leqslant \tau < \mu$. Since $|I| = \mu^+$, we may assume the same σ works for all $g \in I$. But now $\{f\colon$ for some $g \in I$, domain $f_\tau =$ domain $g_\tau \cap D$ for all $\tau \in$ domain g, and $f_\tau(\alpha) =$ $g_\tau(\alpha) \cap Y$ for all $\alpha \in$ domain $f_\tau\}$ has power at most μ, so some single f must work for μ^+ of the g's, and the assertion is established.

A straightforward check shows that any two members of I' are compatible, a contradiction. This proves the first two sentences of Lemma 6.3. The last sentence is left to the reader.

Lemma 6.4. *For any regular $\mu \leqslant \kappa$, $Q(\kappa,\lambda,\mu)$ is μ-complete.*

Proof. Obvious.

Lemma 6.5. *Let G be $Q(\kappa,\lambda,\nu)^{\mathcal{M}}$-generic over $\mathcal{M}$, and assume that μ is a regular cardinal of $\mathcal{M}$ such that $\nu \leqslant \mu < \kappa$. Let*

$$H = \{f|\{\mu'\colon \mu^+ \leqslant \mu' \leqslant \kappa\} \colon f \in G\}\,,$$

$$J = \{f|\{\mu'\colon \nu \leqslant \mu' \leqslant \mu\} \colon f \in G\}\,.$$

Then

(7) $\mathcal{M}[G] = \mathcal{M}[H][J]$,

(8) H *is* $Q(\kappa,\lambda,\mu^+)^{\mathcal{M}}$ *-generic over* $\mathcal{M}$,
(9) J *is* $Q_\mu(\kappa,\lambda,\nu,F)^{\mathcal{M}[H]}$*-generic over* $\mathcal{M}[H]$, *where* $F_\alpha = \bigcup\{f_{\mu^+}(\alpha): f \in H\}$ *for* $\alpha < \lambda$.

Proof. (7) is clear, and (8) is very easily checked. We concentrate on proving (9). First note that by Lemmas 6.4 and 5.3 we have $Q_\mu(\kappa,\lambda,\nu,F)^{\mathcal{M}[H]} \subseteq \mathcal{M}$. Let $\dot{Q}$ be the natural term of the language of forcing (over $\mathcal{M}$ with respect to $Q(\kappa,\lambda,\mu^+)^{\mathcal{M}}$) which denotes $Q_\mu(\kappa,\lambda,\nu,F)$ in $\mathcal{M}[H]$. We work in $\mathcal{M}$. For $f \in Q(\kappa,\lambda,\nu)$, let $f_\mu = f|\{\mu': \nu \leqslant \mu' \leqslant \mu\}$, and let $f^\mu = f - f_\mu$. It is easy to see that $f \Vdash g \in \dot{Q}$ iff $g \cup f \in Q(\kappa,\lambda,\nu)$, $(g \cup f)_\mu = g$ and $(g \cup f)^\mu = f$. Now suppose $f \in H$ and $f \Vdash \dot{D}$ is dense in $\dot{Q}$. Say $f = g^\mu$ for some $g \in G$. It will suffice to prove that $\{f': (f')^\mu \Vdash f'_\mu \in \dot{D}\}$ is dense below g in $Q(\kappa,\lambda,\nu)$. But this is clear. □

Lemma 6.6. *Cofinalities are preserved in* $\mathcal{M}[G]$, *where* G *is* $Q(\kappa,\lambda,\nu)^{\mathcal{M}}$ *-generic over* $\mathcal{M}$.

Proof. It will suffice to prove that every regular cardinal of $\mathcal{M}$ remains a regular cardinal in $\mathcal{M}[G]$. Suppose τ is regular in $\mathcal{M}$ and $\mathrm{cf}\tau = \mu < \tau$ in $\mathcal{M}[G]$. If $\mu < \nu$ then this is impossible by Lemmas 6.4 and 5.3. If $\mu \geqslant \kappa$ then this is impossible by Lemma 5.2 since the partial ordering has the μ^+-chain condition by Lemma 6.3. Hence we have $\nu \leqslant \mu < \kappa$.

Now let H, J and F be as in Lemma 6.5. It is clear from Lemmas 6.4 and 5.3 that $\mathrm{cf}\tau \geqslant \mu^+$ in $\mathcal{M}[H]$. For the same reason (and since GCH holds in $\mathcal{M}$) we have $\mu^{<\mu} = \mu$ in $\mathcal{M}[H]$. Since clearly $|F_\alpha \cap F_\beta| \leqslant \mu$ for $\alpha < \beta < \lambda$, it follows by Lemma 6.3 that $Q_\mu(\kappa,\lambda,\nu,F)$ has the μ^+-chain condition in $\mathcal{M}[H]$. But now by (9) and Lemma 5.2, $(\mathrm{cf}\tau)^{\mathcal{M}[H]}$ remains regular in $\mathcal{M}[H][J] = \mathcal{M}[G]$, so $\mathrm{cf}\tau \geqslant \mu^+$ in $\mathcal{M}[G]$, a contradiction. □

This completes the proof of Theorem 6.1. We remark that under the hypotheses of Theorem 6.1 the following are also true.

(10) if $\nu' < \nu$, *then* $A(\kappa,\kappa^+,\kappa,\nu')$ *is false in* $\mathcal{M}[G]$.
(11) if $\mathrm{cf}\lambda > \nu$ *in* $\mathcal{M}$, *then* $2^\nu = \lambda$ *in* $\mathcal{M}[G]$.

The proofs are left to the reader.

There is at least two approaches to proving that the negation of $A(\aleph_1,\aleph_2,\aleph_1,\aleph_0)$ is consistent with ZFC + $2^{\aleph_0} \geqslant \aleph_2$. The first is by means of the partial orderings $P(\nu,\rho)$ defined in Section 5. It turns out that if G is $P(\aleph_0,\aleph_2)^{\mathcal{M}}$ -generic over $\mathcal{M}$, then $A(\aleph_1,\aleph_2,\aleph_1,\aleph_0)$ fails in $\mathcal{M}[G]$, and this result can be generalized in a natural way. Since the $P(\nu,\rho)$ have already been studied by many people, however, it seems desirable to present the following alternative approach.

Recall that Theorem 2.7 asserts that if $\lambda > \kappa^\nu$ then $A(\kappa,\lambda,\mu,\nu)$ is false. The proof of Theorem 2.7 really uses only the following assertion, which appears to be weaker than $\lambda > \kappa^\nu$:

(12) There is a collection F subsets of κ such that $|F| < \lambda$, $|A| = \nu$ for all $A \in F$, and for every $X \subseteq \kappa$, if $|X| = \mu$ then there is $A \in F$ with $A \subseteq X$.

We denote (12) by $B(\kappa,\lambda,\mu,\nu)$. Thus, to prove that $A(\aleph_1,\aleph_2,\aleph_1,\aleph_0)$ is false, it will suffice to show that $B(\aleph_1,\aleph_2,\aleph_1,\aleph_0)$ is true.

We remark that in the model obtained using the partial ordering $P(\aleph_0,\aleph_2)$, $B(\aleph_1,\aleph_2,\aleph_1,\aleph_0)$ is *not* true.

Let κ and λ be cardinals, and assume that κ is regular. Let $R(\kappa,\lambda)$ be the set of all subsets B of $\lambda \times 2 \times \kappa$ satisfying:

(13) $|B| \leqslant \kappa$,

(14) if $\alpha < \lambda$ and $\beta < \kappa$ then either $(\alpha,0,\beta) \notin B$ or $(\alpha,1,\beta) \notin B$,

(15) for all $\alpha < \lambda$, $\{\beta < \kappa\colon (\alpha,0,\beta) \notin B, (\alpha,1,\beta) \notin B\}$ is closed and unbounded in κ.

Given $B_1,B_2 \in R(\kappa,\lambda)$, let $B_1 \leqslant B_2$ iff $B_2 \subseteq B_1$.

The partial ordering $R(\aleph_0,1)$ has been studied by Prikry and Silver [14]

$\diamondsuit_\kappa$ is the following proposition: There is a sequence $\langle S_\alpha : \alpha < \kappa\rangle$ such that $S_\alpha \subseteq \alpha$ for all $\alpha < \kappa$ and for any $A \subseteq \kappa$, $\{\alpha < \kappa\colon A \cap \alpha = S_\alpha\}$ is stationary in κ.

Jensen [11] has proved that $V = L$ implies $\diamondsuit_\kappa$ for all regular $\kappa > \aleph_0$. Hence in particular the propositions $\diamondsuit_\kappa$ are consistent with the GCH.

Theorem 6.7. *Let κ and λ be cardinals of $\mathfrak{M}$ such that κ is regular and $\kappa \leqslant \lambda$. If κ is not inaccessible in $\mathfrak{M}$, then assume also that $\diamondsuit_\kappa$ holds in $\mathfrak{M}$. If G is $R(\kappa,\lambda)^{\mathfrak{M}}$-generic over $\mathfrak{M}$, then in $\mathfrak{M}[G]$ cofinalities are preserved, $2^\kappa \geqslant \lambda$, and for all $\mu > \kappa$, $B(\mu,\mu^+,\kappa^+,\kappa)$ holds and $A(\mu,\mu^+,\kappa^+,\kappa)$ fails.*

Proof. As we remarked, $B(\mu,\mu^+,\kappa^+,\kappa)$ implies the negation of $A(\mu,\mu^+,\kappa^+,\kappa)$ by the proof of Theorem 2.7. Moreover, it is not difficult to see that if cofinalities are preserved, then $2^\kappa \geqslant \lambda$ in $\mathfrak{M}[G]$: For each $\alpha < \lambda$ let $G_\alpha = \{\beta < \kappa\colon (\exists B \in G)\,(\alpha,0,\beta) \in B\}$. It is clear that $\alpha \neq \alpha'$ implies $G_\alpha \neq G_{\alpha'}$.

Therefore it will suffice to show that every regular cardinal of $\mathfrak{M}$ remains regular in $\mathfrak{M}[G]$, and $B(\mu,\mu^+,\kappa^+,\kappa)$ holds in $\mathfrak{M}[G]$ for all $\mu > \kappa$.

Lemma 6.8. *$R(\kappa,\lambda)$ is κ-complete.*

Proof. The intersection of fewer than κ closed unbounded sets is still closed unbounded. □

For $B \in R(\kappa,\lambda)$, we define the *domain* of B to be

$$\{\alpha < \lambda : (\exists i < 2)(\exists \beta < \kappa)(\alpha, i, \beta) \in B\}.$$

Lemma 6.9. *$R(\kappa,\lambda)$ has the $(2^\kappa)^+$-chain condition.*

Proof. Suppose, on the contrary, that I is a set of pairwise incompatible elements and $|I| = (2^\kappa)^+$. By Lemma 6.2 there are $I' \subseteq I$ and $D \subseteq \lambda$ such that $|D| \leqslant \kappa$, $|I'| = |I|$ and domain $B \cap$ domain $B' = D$ for all $B, B' \in I'$ with $B \neq B'$. But then since $|I'| = (2^\kappa)^+$ there are $I'' \subset I'$ and D' such that $|I''| = |I'|$ and for all $B \in I''$, $B \cap (D \times 2 \times \kappa) = D'$. Now any two members of I'' are compatible, a contradiction. □

Lemma 6.10. *Under the hypotheses of Theorem 6.7, the following is true in $\mathfrak{M}[G]$:*

(16) *Let Y be a set of ordinals with* $\sup Y = \delta$. *If* $\operatorname{cf}\delta \geqslant \kappa^+$ *in $\mathfrak{M}$, then there is $X \in \mathfrak{M}$ such that $X \subseteq Y$ and $|X|^{\mathfrak{M}} = \kappa$.*

Proof. Choose $A \subseteq \kappa$ so that $|A| = |\kappa - A| = \kappa$, and let $\langle A_\alpha : \alpha < \kappa\rangle$ be a sequence (in $\mathfrak{M}$) of pairwise disjoint subsets of $\kappa - A$, each of which has power κ. Let $\dot{Y}$ be a term of the language of forcing which denotes Y in $\mathfrak{M}[G]$, and let $B \in R(\kappa,\lambda)$ be such that

$$B \Vdash \dot{Y} \text{ is a set of ordinals and } \sup \dot{Y} = \delta.$$

Now we work in $\mathfrak{M}$. It will suffice to find $B' \leqslant B$ and X such that $|X| = \kappa$ and $B' \Vdash \check{X} \subseteq \dot{Y}$.

We will construct a sequence $\langle B_\alpha : \alpha < \kappa\rangle$ of members of $R(\kappa,\lambda)$, a sequence $\langle f_\alpha : \alpha < \kappa\rangle$ of functions and a sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of ordinals $< \kappa$ satisfying:

(17) $B_0 = B$,

(18) for all $\alpha < \kappa$, f_α maps $\alpha \cup A \cup \bigcup\{A_\beta : \beta < \alpha\}$ one–one onto domain B_α,

(19) if $\beta < \alpha < \kappa$, then for all $\gamma < \kappa$, $(f_\alpha(\beta), 0, a_\alpha), (f_\alpha(\beta), 1, a_\alpha) \notin B_\gamma$,

(20) if $\alpha < \beta < \kappa$, then $B_\alpha \leqslant B_\beta$, $f_\alpha \subseteq f_\beta$ and $\alpha_\alpha < \alpha_\beta$,

(21) if α is a limit ordinal, then

$$B_\alpha = \bigcup_{\beta<\alpha} B_\beta, \qquad f_\alpha = \bigcup_{\beta<\alpha} f_\beta, \qquad a_\alpha = \sup_{\beta<\alpha} a_\beta .$$

Let f_0 be any function mapping A one–one onto domain B (we may assume $|\text{domain } B| = \kappa$), and let a_0 be arbitrary. The construction for limit ordinals is given completely by (21). Note that a_α as defined by (21) still satisfies (19) because of condition (15) in the definition of $R(\kappa,\lambda)$. The construction for successor ordinals depends on whether or not κ is inaccessible.

Case 1: κ is inaccessible. (Note that this includes the case $\kappa = \aleph_0$.) Given B_α, f_α and a_α, we show how to obtain $B_{\alpha+1}, f_{\alpha+1}$ and $a_{\alpha+1}$. Let $E_\alpha = \{f_\alpha(\beta) \colon \beta < \alpha\}$, and let $\langle D_\beta \colon \beta < \tau_\alpha\rangle$ be an enumeration of all sets D such that

$$B_\alpha \cap (E_\alpha \times 2 \times (\alpha+1)) \subseteq D \subseteq E_\alpha \times 2 \times (\alpha+1) .$$

Since κ is inaccessible, $\tau_\alpha < \kappa$. Now we produce a sequence $\langle C_\beta ; \beta < \tau_\alpha\rangle$ of members of $R(\kappa,\lambda)$ and a sequence $\langle x^\alpha_\beta \colon \beta < \tau_\alpha\rangle$ as follows. Let $C_0 = B_\alpha - (E_\alpha \times 2 \times (\alpha+1))$ and let x^α_0 be undefined. Given C_β, if there are $\overline{B} \in R(\kappa,\lambda)$ and x such that $\overline{B} \leqslant C_\beta$, $\overline{B} \cap (E_\alpha \times 2 \times (\alpha+1)) = D_\beta$, x is distinct from all the $x^{\alpha'}_{\beta'}$ constructed so far (i.e. for $\alpha' < \alpha$ or for $\alpha' = \alpha$ and $\beta' < \beta$), and $\overline{B} \Vdash x \in \dot{Y}$, then let $x^\alpha_\beta = x$ and let $C_{\beta+1} = \overline{B} - (E_\alpha \times 2 \times (\alpha+1))$. Otherwise, let $C_{\beta+1} = C_\beta$ and leave x^α_β undefined. If β is a limit ordinal, let $C_\beta = \bigcup_{\gamma<\beta} C_\gamma$. Now let $B_{\alpha+1}$ be such that

$$|\text{domain } B_{\alpha+1} - \text{domain } B_\alpha| = \kappa,$$

$$B_{\alpha+1} \leqslant (B_\alpha \cap (E_\alpha \times 2 \times (\alpha+1))) \cup \bigcup_{\beta<\tau_\alpha} C_\beta ,$$

$$B_{\alpha+1} \cap (E_\alpha \times 2 \times (\alpha+1)) = B_\alpha \cap (E_\alpha \times 2 \times (\alpha+1)) .$$

Let $f_{\alpha+1}$ be any function satisfying (18) and (20), and let $a_{\alpha+1} > \alpha_\alpha$ be such that (19) is satisfied (which is possible since the intersection of fewer than κ closed unbounded sets is closed unbounded, hence non-empty).

Case 2: κ is not inaccessible. It is easy to see that $\diamondsuit_\kappa$ is equivalent to the following proposition:

(22) there is a sequence $\langle S_\beta \colon \beta < \kappa\rangle$ such that $S_\beta \subseteq \beta \times 2 \times \beta$ for all $\beta < \kappa$ and for all $Z \subseteq \kappa \times 2 \times \kappa$, $\{\beta \colon Z \cap (\beta \times 2 \times \beta) = S_\beta\}$ is stationary in κ.

Given B_α we define $B_{\alpha+1}$ and an associated ordinal x_α as follows. If there exist $\bar{B} \in R(\kappa,\lambda)$ and x such that x is different from all previously defined x_α,

$$\bar{B} \Vdash x \in \dot{Y}, \qquad \bar{B} \leqslant B_\alpha ,$$

$$|\text{domain } \bar{B}\text{-domain } B_\alpha| = \kappa,$$

$$\{(\gamma,i,\gamma') \in \alpha \times 2 \times \alpha : (f_\alpha(\gamma),i,\gamma') \in \bar{B}\} = S_\alpha$$

and for all $\gamma < \alpha$ and $i < 2$,

$$(f_\alpha(\gamma),i,\alpha) \notin \bar{B} ,$$

then let $x_\alpha = x$ and let

$$B_{\alpha+1} = (B_\alpha \cap (E_\alpha \times 2 \times (\alpha+1))) \cup (\bar{B} - (E_\alpha \times 2 \times (\alpha+1))) ,$$

where E_α is defined as in Case 1. Otherwise, let $B_{\alpha+1} \leqslant B_\alpha$ be arbitrary such that

$$B_{\alpha+1} \cap (E_\alpha \times 2 \times (\alpha+1)) = B_\alpha \cap (E_\alpha \times 2 \times (\alpha+1)) ,$$

$$|\text{domain } B_{\alpha+1} - \text{domain } B_\alpha| = \kappa,$$

and leave x_α undefined. Define $f_{\alpha+1}$ and $\tau_{\alpha+1}$ as in Case 1.

This completes the construction.

Now let $B_\kappa = \bigcup_{\alpha<\kappa} B_\alpha$ and $f = \bigcup_{\alpha<\kappa} f_\alpha$. Note that by (19), for any $\alpha \in$ domain B_κ and any $\gamma > f^{-1}(\alpha)$, $(\alpha,0,a_\gamma)$, $(\alpha,1,a_\gamma) \notin B_\kappa$. Hence, since by (20) and (21) the a_γ form a closed unbounded set, there is $B'_\kappa \in R(\kappa,\lambda)$ such that $B_\kappa \subseteq B'_\kappa$. Since cf $\delta \geqslant \kappa^+$, there is some $y < \delta$ such that y is greater than all the x^α_β (if κ is inaccessible) or all the x_α (otherwise). Let $B' \leqslant B'_\kappa$ and x be such that $y < x$ and $B' \Vdash x \in \dot{Y}$. We assert that $\{x' : B' \Vdash x' \in \dot{Y}\}$ has power $\geqslant \kappa$. This will complete the proof.

First assume κ is inaccessible. Fix $\alpha < \kappa$. Then $B' \cap (E_\alpha \times 2 \times (\alpha+1)) = D_\beta$ for some $\beta < \tau_\alpha$. Since $B' \Vdash x \in \dot{Y}$ it is clear that x^α_β was defined. Also, since $B' \leqslant D_\beta \cup C_{\beta+1}$ and $D_\beta \cup C_{\beta+1} \Vdash x^\alpha_\beta \in \dot{Y}$, we have $B' \Vdash x^\alpha_\beta \in \dot{Y}$. Since α was arbitrary, $|\{x' : B' \Vdash x' \in \dot{Y}\}| \geqslant \kappa$.

Now assume κ is accessible. For each $\alpha < \kappa$, let

$$U_\alpha = \{\beta < \kappa : (f(\alpha),0,\beta), \quad (f(\alpha),1,\beta) \notin B'\} .$$

By (15) each U_α is closed unbounded so $U = \{\beta < \kappa : \beta \in U_\gamma$ for all $\gamma < \beta\}$

is also closed unbounded. Let

$$Z = \{(\alpha, i, \beta) \in \kappa \times 2 \times \kappa : (f(\alpha), i, \beta) \in B'\} .$$

By (22), $S = \{\alpha < \kappa : Z \cap (\alpha \times 2 \times \alpha) = S_\alpha\}$ is stationary, so $S \cap U$ is stationary also. Let $\alpha \in S \cap U$ be arbitrary. It is easy to see that since $B' \Vdash x \in \dot{Y}$, x_α was defined. But since

$$B' \leqslant B'' = \{(f(\gamma), i, \gamma') : (\gamma, i, \gamma') \in S_\alpha\} \cup [B_{\alpha+1} - (E_\alpha \times 2 \times (\alpha+1))] ,$$

$$B'' \Vdash x_\alpha \in \dot{Y} ,$$

we have $B' \Vdash x_\alpha \in \dot{Y}$. Hence $|\{x' : B' \Vdash x' \in \dot{Y}\}| \geqslant \kappa$ and the proof is complete. □

Lemma 6.11. *Under the hypotheses of Theorem 6.7, every regular cardinal of $\mathfrak{M}$ remains regular in $\mathfrak{M}$ $[G]$ and $B(\mu, \mu^+, \kappa^+, \kappa)$ holds in $\mathfrak{M}$ $[G]$ for all $\mu > \kappa$.*

Proof. Let ν be regular in $\mathfrak{M}$. If $\nu \leqslant \kappa$ then ν remains regular by Lemmas 6.8 and 5.3. If $\nu \geqslant \kappa^{++}$ then ν remains regular by Lemmas 6.9 and 5.2 (since the GCH is true in $\mathfrak{M}$). Now suppose $\nu = \kappa^+$. If $|\kappa^+|^{\mathfrak{M}[G]} = \kappa$, then there is a cofinal subset Y of $(\kappa^+)^{\mathfrak{M}}$ of order-type κ. By Lemma 6.10, there is $X \subseteq Y$ with $X \in \mathfrak{M}$ and $|X| = \kappa$. Hence X is cofinal with Y so $|\kappa^+|^{\mathfrak{M}} = \kappa$, contradiction.

Now let $\mu > \kappa$ be fixed, and let $Y \subseteq \mu$ have power κ^+ (in $\mathfrak{M}$ $[G]$). We may assume that Y has order-type κ^+. By Lemma 6.10, there is $X \in \mathfrak{M}$ such that $X \subseteq Y$ and $|X| = \kappa$. Note that then X is bounded in Y, hence in μ. But $\{X : X \in \mathfrak{M}, X \subseteq \mu, |X| = \kappa$ and X is bounded in $\mu\}$ has power μ since the GCH is true in $\mathfrak{M}$. Hence $B(\mu, \mu^+, \kappa^+, \kappa)$ holds. □

Remark 6.12. Theorem 6.7 may be improved in the following ways:

(a) If $\mathrm{cf}\lambda > \kappa$ in $\mathfrak{M}$, then $2^\kappa = \lambda$ in $\mathfrak{M}$ $[G]$. If $\mathrm{cf}\lambda \geqslant \kappa^{++}$ then the proof of this is standard, and uses only the fact that $R(\kappa, \lambda)$ has the κ^{++}-chain condition in $\mathfrak{M}$. If $\mathrm{cf}\lambda = \kappa^+$ then the proof is non-trivial, and requires the following observation:

(23) If $B \Vdash \tau : \kappa \to 2$, then there is $B' \leqslant B$ such that for all $\alpha < \kappa$ and all $C \leqslant B'$, if $C \Vdash \tau(\alpha) = i$ then there is $D \subseteq C$ such that $|D| < \kappa$ and $B' \cup D \Vdash \tau(\alpha) = i$.

The proof of (23) is similar to the proof of Lemma 6.10.

(b) The fact that κ^+ remains a cardinal in $\mathfrak{M}$ $[G]$ may be proved using the following instead of Lemma 6.10, when κ is inaccessible.

(24) If $B \Vdash \tau: \kappa \to \mathfrak{M}$, then there are $B' \leqslant B$ and a function $t \in \mathfrak{M}$ with domain κ such that $t(\alpha) \subseteq {}^{\alpha}\mathfrak{M}$ and $|t(\alpha)| < \kappa$ for all $\alpha < \kappa$, and $B' \Vdash (\forall\alpha < \kappa)\, \tau|\alpha \in t(\alpha)$.

The proof of (24) is also similar to the proof of Lemma 6.10. Moreover, (24) yields some additional information. For example, using (24) one can see that for any function $f: \kappa \to \kappa$ in $\mathfrak{M}[G]$, there is a function $g: \kappa \to \kappa$ in $\mathfrak{M}$ such that $g(\alpha) > f(\alpha)$ for all $\alpha < \kappa$.

(c) The property $B(\kappa,\lambda,\mu,\nu)$ may be generalized by replacing κ and ν by arbitrary ordinals α and β. $B(\alpha,\lambda,\mu,\beta)$ would say that there is a family F of subsets of α such that $|F| < \lambda$, every $A \in F$ has order-type β, and for every $X \subseteq \alpha$, if $|X| = \mu$ then there is $A \in F$ with $A \subseteq X$. By complicating the proof of Lemma 6.10, one could show the following:

(25) Let Z be a set of ordinals well-ordered by $\prec$ in $\mathfrak{M}$, and let $\beta < \kappa^+$. If $Y \in \mathfrak{M}[G]$ is a subset of Z which has order-type κ^+ with respect to $\prec$, then there is $X \subseteq Y$ such that $X \in \mathfrak{M}$ and X has order-type β with respect to $\prec$.

Using (25), it is now easy to see that in $\mathfrak{M}[G]$, $B(\alpha,\mu^+,\kappa^+,\beta)$ holds whenever $\mu > \kappa$, $\mu \leqslant \alpha < \mu^+$ and $\beta < \kappa^+$. The proof runs as follows. Given μ,α and β, fix a well-ordering $\prec$ of μ in type α. Let $Y \subseteq \mu$ have type κ^+ with respect to $\prec$. We claim that there is $X \subseteq Y$ such that $X \in \mathfrak{M}$, X has type β with respect to Y, and X is bounded in μ with respect to the *usual* ordering of μ. Then the proof will be complete since the GCH holds in $\mathfrak{M}$. If $\mathrm{cf}\mu \geqslant \kappa^+$, then the claim is obvious. If $\mathrm{cf}\mu \leqslant \kappa$, then there is $Y' \subseteq Y$ such that Y' has type κ^+ with respect to $\prec$, and Y is bounded in μ with respect to the usual ordering. Applying (25) to Y' proves the claim.

This remark will be used in the next section.

7.

There appears to be a close connection between the almost-disjoint-set property and parts of the partition calculus. In Section 5, we used a partition theorem to prove independence results about almost-disjoint sets. Here we reverse the process by applying the methods of Section 6 to obtain independence results in the partition calculus.

For notation see Section 1. A discussion of the problems considered here may be found in [6].

Theorem 7.1. *Let $\kappa,\lambda,\mathfrak{M}$ and G be as in Theorem 6.7. Then the following are true in $\mathfrak{M}[G]$:*

(a) *For any cardinals μ,ν and ρ, and any ordinal β_0, if* $\mathrm{cf}\nu > \kappa$, $\mathrm{cf}\mu > \nu$, $\rho < \mathrm{cf}\nu$ *and* $\beta_0 < \kappa^+$, *then*

$$\binom{\mu}{\nu} \to \binom{\mu}{\nu}\binom{\mu}{\beta_0}_{\rho}^{1,1} .$$

Hence in particular

$$\binom{\mu}{\nu} \to \binom{\mu\ \ \mu}{\nu\ \ \beta_0}^{1,1} , \qquad \binom{\mu}{\nu} \to \binom{\mu}{\beta_0}_{\rho}^{1,1} .$$

(b) *If $\mu \geqslant \kappa^{++}$ and μ is regular, then for every $\beta_0 < \kappa^+$,*

$$\mu \to (\mu, [\beta_0,\mu])^2 .$$

In fact, if $[\mu]^2 = P_0 \cup P_1$, then either there is a stationary set $A \subseteq \mu$ with $[A]^2 \subseteq P_0$, or there are sets $A,B \subseteq \mu$ such that A has order-type β_0, B is stationary and $[A,B] \subseteq P_1$. The latter statement could be written

$$\mu \to (\textit{stationary}\ \mu,\ [\beta_0,\ \textit{stationary}\ \mu])^2 .$$

Proof. (a) By Remark 6.12(c), there is a family F of subsets of ν such that $|F| = \nu$, every member of F has order-type β_0, and every subset of ν of power κ^+ contains a member of F. Let $f: \mu \times \nu \to 1 + \rho$.

Case 1: $|\{\alpha < \mu: |\{\beta < \nu: f(\alpha,\beta) \neq 0\}| < \mathrm{cf}\nu\}| = \mu$. Then since $\mathrm{cf}\mu > \nu$ there is a set $A \subseteq \mu$ and an ordinal $\gamma < \nu$ such that $|A| = \mu$ and for all $\alpha \in A$ and all $\beta > \gamma$, $f(\alpha,\beta) = 0$. Hence f is constant on $A \times (\nu - \gamma)$.

Case 2: Otherwise. Since $\mathrm{cf}\mu > \rho$ and $\mathrm{cf}\nu > \rho,\kappa$, there is a set $A \subseteq \mu$ and an ordinal $\delta < \rho$ such that $|A| = \mu$ and for all $\alpha \in A$,

$$|\{\beta < \nu: f(\alpha,\beta) = 1 + \delta\}| \geqslant \kappa^+ .$$

Hence for each $\alpha \in A$ there is a set $X_\alpha \in F$ with $X_\alpha \subseteq \{\beta < \nu: f(\alpha,\beta) = 1 + \delta\}$. Since $\mathrm{cf}\mu > \nu$, there are $A' \subseteq A$ and $X \in F$ with $|A'| = \mu$ and $X_\alpha = X$ for all $\alpha \in A'$. But then f is constant on $A' \times X$.

(b) Let $g: [\mu]^2 \to 2$. First assume $\mu > \kappa^{++}$. Then $C = \{\alpha < \mu: \mathrm{cf}\alpha = \kappa^{++}\}$ is stationary in μ.

Case 1: There is a stationary set $S \subseteq S$ such that for all $\alpha \in S$, $\{\beta < \alpha: g(\{\alpha,\beta\}) = 1\}$ is bounded in α. Define $h: S \to \mu$ by

$$h(\alpha) = \sup(\{\beta < \alpha: g(\{\alpha,\beta\}) = 1\}) + 1 .$$

By a well-known theorem of Fodor [9], there is then a stationary set $S' \subseteq S$ and an ordinal γ so that $h(\alpha) = \gamma$ for all $\alpha \in S'$. But then S' is stationary and homogeneous for g.

Case 2: Otherwise. For each $\alpha \in S$ let $h'(\alpha)$ be the supremum of the first κ^+ members of $\{\beta < \alpha : f(\{\alpha,\beta\}) = 1\}$. Since $h'(\alpha) < \alpha$ for all but a non-stationary subset of S, we may apply Fodor's theorem again to obtain stationary $S' \subseteq S$ and an ordinal γ so that $h'(\alpha) = \gamma$ for all $\alpha \in S'$. By Remark 6.12(c), there is a family F of subsets of γ such that $|F| = |\gamma|$, every $A \in F$ has order-type β_0, and every subset of γ of cardinality κ^+ contains a member of F. Hence for each $\alpha \in S'$ there is $X_\alpha \in F$ such that $f(\{\alpha,\beta\}) = 1$ for all $\beta \in X_\alpha$. Since $|F| = |\gamma|$ there are $S'' \subseteq S'$ and X such that S'' is stationary and $X_\alpha = X$ for all $\alpha \in S''$. But then $f(x) = 1$ for all $x \in [X, S'']$.

Now assume $\mu = \kappa^{++}$. Let $C = \{\alpha < \mu : \mathrm{cf}\alpha = \kappa^+\}$. Case 1 goes through exactly as before, so we concentrate on Case 2. By Remark 6.12(c), every subset of μ of power κ^+ contains a subset of type β_0 *which lies in* $\mathfrak{M}$. For each $\alpha \in S$ let X_α be such a subset of $\{\beta < \alpha : f(\{\alpha,\beta\}) = 1\}$ (provided the latter set has cardinality κ^+) and let $h''(\alpha) = \sup X_\alpha$. Then $h''(\alpha) < \alpha$ for all but a non-stationary subset of S. Let $S' \subseteq S$ and γ be such that S' is stationary and $h''(\alpha) = \gamma$ for all $\alpha \in S'$. Since $X_\alpha \in \mathfrak{M}$ for all $\alpha \in S'$ there are at most $(|\gamma|^\kappa)^{\mathfrak{M}} = \kappa^+$ such sets X_α. Hence there is $S'' \subseteq S'$ and X such that S'' is stationary and $X_\alpha = X$ for all $\alpha \in S''$. Then $f(x) = 1$ for all $x \in [X, S'']$. □

Consider the special case of Theorem 7.1 in which $\kappa = \aleph_0$, λ is any regular cardinal $\geqslant \aleph_2$, $\mu = \lambda$ and $\nu = \aleph_1$. Then in $\mathfrak{M}[G]$,

$$2^{\aleph_0} = \lambda, \qquad \binom{2^{\aleph_0}}{\aleph_1} \to \left(\binom{2^{\aleph_0}}{\aleph_1}\binom{2^{\aleph_0}}{\alpha}_{\aleph_0}\right)^{1,1} \quad \text{for all } \alpha < \omega_1,$$

$$2^{\aleph_0} \to (2^{\aleph_0}, [\alpha, 2^{\aleph_0}])^2 \quad \text{for all } \alpha < \omega_1 .$$

It would be very interesting to know whether in this situation it is also true that $2^{\aleph_0} \to (2^{\aleph_0}, \alpha)^2$ for all $\alpha < \omega_1$.

It turns out that the existence of strong counterexamples to the partition relations of Theorem 7.1 is also consistent with ZFC + $2^{\aleph_0} \geqslant \aleph_2$. For this we use the method of Theorem 6.1.

Let $\mathfrak{M}$ be a countable transitive model of ZFC + GCH, and let κ and ν be regular cardinals in $\mathfrak{M}$ with $\nu \leqslant \kappa$. Let $Q = Q_\kappa(\kappa^+, \kappa^+, \nu, F)^{\mathfrak{M}}$, where $F = \langle \alpha : \alpha < \kappa^+ \rangle$.

Theorem 7.2. *If G is Q-generic over $\mathcal{M}$, then in $\mathcal{M}[G]$ cofinalities are preserved, $2^\nu = \kappa^+$ and*

$$\kappa^+ \not\to (\kappa^+, [\nu,2])^2 .$$

Proof. The facts that cofinalities are preserved and $2^\nu = \kappa^+$ are handled as in Theorem 6.1.

Recall that, if $f \in Q$, then $f = \langle f_\mu : \nu \leqslant \mu \leqslant \kappa, \mu \text{ regular}\rangle$. For each $\alpha < \kappa^+$, let

$$A_\alpha = \bigcup\{f_\kappa(\alpha) : f \in G\}, \qquad B_\alpha = \bigcup\{f_\nu(\alpha) : f \in G\} .$$

Now let $T \subseteq \kappa^+$ have power κ in $\mathcal{M}[G]$. We claim that either

(1) there is a set $x \subseteq \kappa^+$ such that $|x| < \kappa$ and $|T - \bigcup_{\alpha \in x} A_\alpha| < \kappa$, or

(2) there is $\alpha < \kappa^+$ such that for all $\beta \geqslant \alpha$, $B_\beta \cap T \neq 0$.

Let $\dot{T}$ be a term denoting T in $\mathcal{M}[G]$, and let $\alpha_0 < \kappa^+$ and $p \in G$ be such that p forces $|\dot{T}| = \kappa$, $\dot{T} \subseteq \alpha_0$, and the negation of (1) with T replaced by $\dot{T}$ and the A_α replaced by appropriate terms. Now we work in $\mathcal{M}$.

We define a sequence $\langle \alpha_\xi : \xi < \kappa\rangle$ of ordinals $< \kappa^+$ as follows. α_0 is already defined. If $\xi > 0$ and α_η has been obtained for each $\eta < \xi$, then for all $\alpha < \sup_{\eta<\xi} \alpha_\eta$ and all $x \subseteq \sup_{\eta<\xi} \alpha_\eta$ with $|x| < \kappa$, let $X_{\alpha x}$ be a maximal pairwise incompatible set of conditions $q \leqslant p$ such that

$$q \Vdash \alpha \in \dot{T} - \bigcup_{\beta \in x} A_\beta .$$

Note that if $q \Vdash \alpha \notin \bigcup_{\beta \in x} A_\beta$, then $\alpha \notin \bigcup\{q_\kappa(\beta) : \beta \in x\}$. Since Q has the κ^+-chain condition by Lemma 6.3, we have $|X_{\alpha x}| \leqslant \kappa$ for all α. Hence there exists $\alpha_\xi < \kappa^+$ such that $\sup_{\eta<\xi} \alpha_\eta < \alpha_\xi$ and

$$\sup\,(\bigcup\{\text{domain } f_\kappa : f \in \bigcup\{X_{\alpha x} : \alpha < \sup_{\eta<\xi} \alpha_\eta,$$
$$x \subseteq \sup_{\eta<\xi} \alpha_\eta \text{ and } |x| < \kappa\}\}) < \alpha_\xi .$$

Let $\alpha' = \sup_{\xi<\kappa} \alpha_\xi$. We claim p forces (2) with $\alpha = \alpha'$.

Let $f \leqslant p$. Let $x = \text{domain } f_\kappa \cap \alpha'$ and let $y = \bigcup\{f_\kappa(\beta) : \beta \in \text{domain } f_\kappa\}$. Since $|x|, |y| < \kappa$ there are $f' \leqslant f$ and $\gamma \in \alpha_0 - y$ such that $f' \Vdash \gamma \in \dot{T} - \bigcup_{\beta \in x} A_\beta$. Since $X_{\gamma x}$ was maximal, f' is compatible with some $q \in X_{\gamma x}$. Hence f is also compatible with q, and $\gamma \notin \bigcup\{q_\kappa(\beta) : \beta \in x\}$. Now let $\beta > \alpha'$ be fixed and define g by domain g_μ = domain $q_\mu \cup$ domain $f_\mu \cup \{\beta\}$ and $g_\mu(\alpha) = q_\mu(\alpha) \cup f_\mu(\alpha)$ if $\alpha \in \text{domain } g_\mu$ and $\alpha \neq \beta$, and

$g_\mu(\beta) = f_\mu(\beta) \cup \{\gamma\}$. It is easy to see that $g \leqslant f, q$ and $g \Vdash \gamma \in B_\beta \cap \dot{T}$. This proves the claim.

Now, working in $\mathcal{M}[G]$, define a partition $[\kappa^+]^2 = P_0 \cup P_1$ as follows. If $\alpha < \beta < \kappa^+$ and $\alpha \in B_\beta$, then put $\{\alpha,\beta\} \in P_1$; otherwise put $\{\alpha,\beta\} \in P_0$. It is clear that if $\alpha \neq \beta$ then $|B_\alpha \cap B_\beta| < \nu$. Therefore there are no sets $X, Y \subseteq \kappa^+$ with $|X| = \nu$, $|Y| = 2$ and $[X, Y] \subseteq P_1$. Suppose $Z \subseteq \kappa^+$ and $|Z| = \kappa^+$. We claim $[Z]^2 \not\subseteq P_0$. Assume not. Let $T \subseteq Z$ with $|T| = \kappa$. If (2) holds for T, then clearly $[Z]^2 \not\subseteq P_0$. Hence for all $T \subseteq Z$ with $|T| = \kappa$, (1) holds. We construct sequences $\langle T_\alpha : \alpha < \kappa\rangle$ and $\langle x_\alpha : \alpha < \kappa\rangle$ as follows: Given T_β and x_β for all $\beta < \alpha$, let

$$T_\alpha \subseteq Z - \sup \bigcup_{\beta<\alpha} T_\beta \cup \bigcup_{\beta<\alpha} x_\beta$$

have power κ, and let x_α be such that $|x_\alpha| < \kappa$ and $|T_\alpha - \bigcup_{\gamma\in x_\alpha} A_\gamma| < \kappa$. Let $T = \bigcup T_\alpha$ and let x be such that $|x| < \kappa$ and $|T - \bigcup_{\gamma \in x} A_\gamma| < \kappa$. Since κ is regular there is $\gamma \in x$ such that $|\{\alpha : |T_\alpha \cap A_\gamma| = \kappa\}| = \kappa$. Clearly $\gamma \notin \bigcup_{\alpha<\kappa} x_\alpha$. Fix α so that $|T_\alpha \cap A_\gamma| = \kappa$. Then there is $\delta \in x_\alpha$ such that $|T_\alpha \cap A_\gamma \cap A_\delta| = \kappa$, so in particular $|A_\gamma \cap A_\delta| = \kappa$, contradicting the obvious fact that $|A_\alpha \cap A_\beta| < \kappa$ whenever $\alpha \neq \beta$. Hence $\kappa^+ \not\to (\kappa^+, [\nu,2])^2$ in $\mathcal{M}[G]$. □

Note that if $\nu = \aleph_0$ in Theorem 7.2, then in $\mathcal{M}[G]$ $2^{\aleph_0} = \kappa^+$ and $2^{\aleph_0} \not\to (2^{\aleph_0}, [\aleph_0,2])^2$. Hence in particular $2^{\aleph_0} \not\to (2^{\aleph_0}, \omega + 2)^2$. We do not know if it is consistent for $2^{\aleph_0}$ to be weakly inaccessible and $2^{\aleph_0} \not\to (2^{\aleph_0}, \omega + 2)^2$.

The consistency of $2^{\aleph_0} \not\to (2^{\aleph_0}, [\aleph_0,2])^2$ with $2^{\aleph_0} = \aleph_2$ was first obtained by Laver, using a different method (see [6]). It is not known if Laver's approach generalizes to the case $2^{\aleph_0} = \aleph_3$.

Next we deal with the consistency of propositions of the form $\binom{\lambda}{\kappa^+} \not\to \binom{2}{\nu}^{1,1}_{\kappa}$. To obtain the consistency of $\binom{\aleph_2}{\aleph_1} \not\to \binom{2}{\aleph_0}^{1,1}_{\aleph_0}$, for example, we proceed as follows. First, via an $\aleph_1$-complete, $\aleph_2$-chain condition partial ordering, we get a generic sequence $\langle\langle X^\alpha_n : n < \omega\rangle : \alpha < \omega_2\rangle$ of partitions of ω_1 with the property that $\alpha \neq \alpha'$ implies $|X^\alpha_n \cap X^{\alpha'}_n| \leqslant \aleph_0$ for all n. Next, via a countable chain condition partial ordering, we obtain generic partitions $\langle X^\alpha_{nm} : m < \omega\rangle$ of each X^α_n with the property that $\alpha \neq \alpha'$ implies $|X^\alpha_{nm} \cap X^{\alpha'}_{nm}| < \aleph_0$ for all n and m. This is akin the "thinning out" procedure of Theorem 6.1. Finally, define $f: \omega_2 \times \omega_1 \to \omega \times \omega$ by $f(\alpha,\beta) = (n, m)$ iff $\beta \in X^\alpha_{nm}$. This f yields a counterexample to $\binom{\aleph_2}{\aleph_1} \to \binom{2}{\aleph_0}^{1,1}_{\aleph_0}$.

Just as in Theorem 6.1, the general partial ordering involves iterating this "thinning out" procedure.

Let $\mathcal{M}$ be a countable transitive model of ZFC + GCH and let κ, λ and ν be cardinals of $\mathcal{M}$ such that κ and ν are regular and $\nu \leqslant \kappa < \lambda$. We define a partial ordering S in $\mathcal{M}$. For each regular cardinal μ such that $\nu \leqslant \mu \leqslant \kappa^+$, let K_μ be a partition of κ into κ non-empty sets such that if $\mu < \mu'$ and $Y \in K_{\mu'}$ then $\{X \in K_\mu : X \subseteq Y\}$ is a partition of Y. Thus K_μ refines $K_{\mu'}$. Moreover, we assume that for all $Y \in K_{\mu^+}$, $|\{X \in K_\mu : X \subseteq Y\}| = \mu$, and that $K_\nu = \{\{\alpha\} : \alpha < \kappa\}$.

Now if μ is regular and $\nu \leqslant \mu \leqslant \kappa$, let $S(\mu)$ be the set of all functions f such that

(3) domain $f \subseteq \lambda \times K_\mu$ and $|\text{domain } f| < \mu$.

(4) if $(\alpha, X) \in$ domain f, then $f(\alpha, X) \subseteq \kappa^+$ and $|f(\alpha, X)| < \mu$

(5) if $(\alpha, X), (\alpha, X') \in$ domain f and $X \neq X'$, then $f(\alpha, X) \cap f(\alpha, X') = 0$.

For $f, g \in S(\mu)$, let $f \leqslant_\mu g$ iff domain $g \subseteq$ domain f, $g(\alpha, X) \subseteq f(\alpha, X)$ for all $(\alpha, X) \in$ domain g, and $g(\alpha, X) \cap g(\alpha', X) = f(\alpha, X) \cap f(\alpha', X)$ whenever $(\alpha, X), (\alpha', X) \in$ domain g and $\alpha \neq \alpha'$.

If $\mu = \kappa^+$ then $S(\mu)$ is the set of all functions satisfying (3), (4), (5) and:

(6) there is an ordinal $\alpha < \kappa^+$ such that for all (β, X), if $(\beta, X) \in$ domain f then $\{\beta\} \times K_\mu \subseteq$ domain f and $\bigcup_{X \in K_\mu} f(\beta, X) = \alpha$.

$\leqslant_\mu$ is defined as before.

Now let S be the set of all $f \in \prod_{\nu \leqslant \mu \leqslant \kappa^+} S(\mu)$ such that if $\mu < \mu'$, $(\alpha, X) \in$ domain f_μ, $Y \in K_{\mu'}$ and $X \subseteq Y$, then $(\alpha, Y) \in$ domain $f_{\mu'}$ and $f_\mu(\alpha, X) \subseteq f_{\mu'}(\alpha, Y)$. For $f, g \in S$, let $f \leqslant g$ iff $f_\mu \leqslant_\mu g_\mu$ for all μ.

Theorem 7.3. *If G is S-generic over $\mathcal{M}$, then in $\mathcal{M}[G]$ cofinalities are preserved and $\binom{\lambda}{\kappa^+} \not\to \binom{2}{\nu}^{1,1}_\kappa$. Moreover, if $\mathrm{cf}\lambda > \nu$ in $\mathcal{M}$, then $2^\nu = \lambda$ in $\mathcal{M}[G]$.*

Proof. Define $h: \lambda \times \kappa^+ \to \kappa$ by $h(\alpha, \beta) = \gamma$ iff there if $f \in G$ such that $\beta \in f_\nu(\alpha, \{\gamma\})$. It is easy to see that h is a counterexample to $\binom{\lambda}{\kappa^+} \to \binom{2}{\nu}^{1,1}$ The rest of the proof is an easy variation of the proof of Theorem 6.1. $\square$

Of course, Theorem 7.3 yields the consistency of $\binom{2^{\aleph_0}}{\aleph_1} \not\to \binom{2}{\aleph_0}^{1,1}_{\aleph_0}$ with $2^{\aleph_0} = \aleph_2$. Note also that if $\lambda = \kappa^{++}$ in Theorem 7.3 and G' is $S(\kappa^+)^{\mathcal{M}}$-generic over $\mathcal{M}$, then in $\mathcal{M}[G']$ the GCH is still true but $\binom{\kappa^{++}}{\kappa^+} \not\to \binom{2}{\kappa^+}^{1,1}_\kappa$.

The following theorem does not use the method of Theorem 6.1, but we include it for completeness.

Theorem 7.4. *Let $\mathcal{M}$ be a countable transitive model of* ZFC + GCH. *Let ν and ρ be cardinals of $\mathcal{M}$ such that ν is regualr and $\nu < \rho$. If G is $P(\nu,\rho)^{\mathcal{M}}$-generic over $\mathcal{M}$, then, in $\mathcal{M}[G]$,*

$$\binom{\rho}{\rho} \not\to \binom{\nu^+}{\nu}^{1,1}_{2}$$

The proof is quite easy, and is left to the reader.

References

[1] J.E. Baumgartner, On the cardinality of dense subsets of linear orderings I, Notices Am. Math Soc. 15 (1968) 935.

[2] J.E. Baumgartner, Results and independence proofs in combinatorial set theory, Ph.D. dissertation, University of California, Berkeley, 1970.

[3] J.L. Bell and A.B. Slomson, Models and Ultraproducts (North-Holland, Amsterdam, 1970).

[4] C.C. Chang, Ultraproducts and other methods of constructing models, in: J.N. Crossley, ed., Sets, Models and Recursion Theory, (North-Holland, Amsterdam, 1967) pp. 85–121.

[5] W.B. Easton, Powers of regular cardianls, Ann. Math. Logic 1 (1970) 139–178.

[6] P. Erdos and A. Hajnal, Unsolved and solved problems in set theory, Proc. Symp. in Pure Math. Vol. 15 (Am. Math. Soc., Providence, RI, 1975) 269–287.

[7] P. Erdos and R. Rado, Intersection theorems for systems of sets, J. London Math. Soc. 35 (1960) 85–90.

[8] P. Erdos and R. Rado, A partition calculus in set theory, Bull. Am. Math. Soc. 62 (1956) 427–489.

[9] G. Fodor, Eine Bermerkung zur Theorie der regressiven Funktionen, Acta Sci. Math. 17 (1956) 139–142.

[10] T. Jech, Lectures in Set Theory (Springer, Berlin, 1971).

[11] R. Jensen, The fine structure of the constructible hierarchy, Ann Math. Logic 4 (1972) 229–308.

[12] A. Levy, Definability in axiomatic set theory I, in: Y. Bar-Hillel, ed., Proc. 1964 Int. Congress for Logic, Methodology and the Philosophy of Science, Jerusalem (North-Holland, Amsterdam, 1965) pp. 127–151.

[13] J. Malitz, the Hanf number for complete $L_{\omega_1,\omega}$ sentences, in: J. Barwise, ed., The Syntax and Semantics of Infinitary Languages, (Springer, Berlin, 1968) pp. 166–181.

[14] A.R.D. Mathias, A survey of recent results in set theory, preprint.

[15] E. Michael, A note on intersections, Proc. Am. Math. Soc. 13 (1962) 281–283.

[16] W. Mitchell, Aronszajn trees and the independence of the transfer property, Ann. Math. Logic 5 (1972) 21–46.

[17] S. Shelah, The number of non-isomorphic models of an unstable first-order theory, Israel J. Math. 9 (1971) 473–487.

[18] W. Sierpinski, Sur une decomposition d'ensembles, Monatsh. Math. Physik 35 (1928) 239–242.

[19] R.M. Solovay, A model of set theory in which every set of reals is Lebesgue measurable, Ann. Math. 92 (1970) 1–56.

[20] A Tarski, Sur la decomposition des ensembles en sous-ensembles presque disjoints, Fund. Math. 12 (1928) 188–205.